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(continued after index)

Stig Larsson · Vidar Thomée

Partial Differential Equations with Numerical Methods

Springer

Stig Larsson
Vidar Thomée
Mathematical Sciences
Chalmers University of Technology
and University of Gothenburg
412 96 Göteborg
Sweden
stig@chalmers.se
thomee@chalmers.se

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Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series Texts in Applied Mathematics (TAM).

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and to encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the Applied Mathematical Sciences (AMS) series, which will focus on advanced textbooks and research-level monographs.

Pasadena, California — J.E. Marsden
New York, New York — L. Sirovich
College Park, Maryland — S.S. Antman

Preface

Our purpose in this book is to give an elementary, relatively short, and hopefully readable account of the basic types of linear partial differential equations and their properties, together with the most commonly used methods for their numerical solution. Our approach is to integrate the mathematical analysis of the differential equations with the corresponding numerical analysis. For the mathematician interested in partial differential equations or the person using such equations in the modelling of physical problems, it is important to realize that numerical methods are normally needed to find actual values of the solutions, and for the numerical analyst it is essential to be aware that numerical methods can only be designed, analyzed, and understood with sufficient knowledge of the theory of the differential equations, using discrete analogues of properties of these.

In our presentation we study the three major types of linear partial differential equations, namely elliptic, parabolic, and hyperbolic equations, and for each of these types of equations the text contains three chapters. In the first of these we introduce basic mathematical properties of the differential equation, and discuss existence, uniqueness, stability, and regularity of solutions of the various boundary value problems, and the remaining two chapters are devoted to the most important and widely used classes of numerical methods, namely finite difference methods and finite element methods.

Historically, finite difference methods were the first to be developed and applied. These are normally defined by looking for an approximate solution on a uniform mesh of points and by replacing the derivatives in the differential equation by difference quotients at the mesh-points. Finite element methods are based instead on variational formulations of the differential equations and determine approximate solutions that are piecewise polynomials on some partition of the domain under consideration. The former method is somewhat restricted by the difficulty of adapting the mesh to a general domain whereas the latter is more naturally suited for a general geometry. Finite element methods have become most popular for elliptic and also for parabolic problems, whereas for hyperbolic equations the finite difference method continues to dominate. In spite of the somewhat different philosophy underlying the two classes it is more reasonable in our view to consider the latter as further developments of the former rather than as competitors, and we feel that the practitioner of differential equations should be familiar with both.

To make the presentation more easily accessible, the elliptic chapters are preceded by a chapter about the two-point boundary value problem for a second order ordinary differential equation, and those on parabolic and hyperbolic evolution equations by a short chapter about the initial value problem for a system of ordinary differential equations. We also include a chapter about eigenvalue problems and eigenfunction expansion, which is an important tool in the analysis of partial differential equations. There we also give some simple examples of numerical solution of eigenvalue problems.

The last chapter provides a short survey of other classes of numerical methods of importance, namely collocation methods, finite volume methods, spectral methods, and boundary element methods.

The presentation does not presume a deep knowledge of mathematical and functional analysis. In an appendix we collect some of the basic material that we need in these areas, mostly without proofs, such as elements of abstract linear spaces and function spaces, in particular Sobolev spaces, together with basic facts about Fourier transforms. In the implementation of numerical methods it will normally be necessary to solve large systems of linear algebraic equations, and these generally have to be solved by iterative methods. In a second appendix we therefore include an orientation about such methods.

Our purpose has thus been to cover a rather wide variety of topics, notions, and ideas, rather than to expound on the most general and far-reaching results or to go deeply into any one type of application. In the problem sections, which end the various chapters, we sometimes ask the reader to prove some results which are only stated in the text, and also to further develop some of the ideas presented. In some problems we propose testing some of the numerical methods on the computer, assuming that MATLAB or some similar software is available. At the end of the book we list a number of standard references where more material and more detail can be found, including issues concerned with implementation of the numerical methods.

This book has developed from courses that we have given over a rather long period of time at Chalmers University of Technology and Göteborg University originally for third year engineering students but later also in beginning graduate courses for applied mathematics students. We would like to thank the many students in these courses for the opportunities for us to test our ideas.

Göteborg, *Stig Larsson*
January, 2003 *Vidar Thomée*

In the second printing 2005 we have corrected several misprints and minor inadequacies, and added a few problems. *SL & VT*

Contents

1 Introduction 1
- 1.1 Background 1
- 1.2 Notation and Mathematical Preliminaries 4
- 1.3 Physical Derivation of the Heat Equation 7
- 1.4 Problems 12

2 A Two-Point Boundary Value Problem 15
- 2.1 The Maximum Principle 15
- 2.2 Green's Function 18
- 2.3 Variational Formulation 20
- 2.4 Problems 23

3 Elliptic Equations 25
- 3.1 Preliminaries 25
- 3.2 A Maximum Principle 26
- 3.3 Dirichlet's Problem for a Disc. Poisson's Integral 28
- 3.4 Fundamental Solutions. Green's Function 30
- 3.5 Variational Formulation of the Dirichlet Problem 32
- 3.6 A Neumann Problem 35
- 3.7 Regularity 37
- 3.8 Problems 38

4 Finite Difference Methods for Elliptic Equations 43
- 4.1 A Two-Point Boundary Value Problem 43
- 4.2 Poisson's Equation 46
- 4.3 Problems 49

5 Finite Element Methods for Elliptic Equations 51
- 5.1 A Two-Point Boundary Value Problem 51
- 5.2 A Model Problem in the Plane 57
- 5.3 Some Facts from Approximation Theory 60
- 5.4 Error Estimates 63
- 5.5 An A Posteriori Error Estimate 66
- 5.6 Numerical Integration 67
- 5.7 A Mixed Finite Element Method 71
- 5.8 Problems 73

6 The Elliptic Eigenvalue Problem . . . 77
6.1 Eigenfunction Expansions . . . 77
6.2 Numerical Solution of the Eigenvalue Problem . . . 88
6.3 Problems . . . 93

7 Initial-Value Problems for ODEs . . . 95
7.1 The Initial Value Problem for a Linear System . . . 95
7.2 Numerical Solution of ODEs . . . 101
7.3 Problems . . . 106

8 Parabolic Equations . . . 109
8.1 The Pure Initial Value Problem . . . 109
8.2 Solution by Eigenfunction Expansion . . . 114
8.3 Variational Formulation. Energy Estimates . . . 120
8.4 A Maximum Principle . . . 122
8.5 Problems . . . 124

9 Finite Difference Methods for Parabolic Problems . . . 129
9.1 The Pure Initial Value Problem . . . 129
9.2 The Mixed Initial-Boundary Value Problem . . . 138
9.3 Problems . . . 146

10 The Finite Element Method for a Parabolic Problem . . . 149
10.1 The Semidiscrete Galerkin Finite Element Method . . . 149
10.2 Some Completely Discrete Schemes . . . 156
10.3 Problems . . . 159

11 Hyperbolic Equations . . . 163
11.1 Characteristic Directions and Surfaces . . . 163
11.2 The Wave Equation . . . 166
11.3 First Order Scalar Equations . . . 169
11.4 Symmetric Hyperbolic Systems . . . 173
11.5 Problems . . . 181

12 Finite Difference Methods for Hyperbolic Equations . . . 185
12.1 First Order Scalar Equations . . . 185
12.2 Symmetric Hyperbolic Systems . . . 192
12.3 The Wendroff Box Scheme . . . 196
12.4 Problems . . . 198

13 The Finite Element Method for Hyperbolic Equations . . . 201
13.1 The Wave Equation . . . 201
13.2 First Order Hyperbolic Equations . . . 205
13.3 Problems . . . 216

14 Some Other Classes of Numerical Methods 217
14.1 Collocation methods 217
14.2 Spectral Methods 218
14.3 Finite Volume Methods 219
14.4 Boundary Element Methods 221
14.5 Problems 223

A Some Tools from Mathematical Analysis 225
A.1 Abstract Linear Spaces 225
A.2 Function Spaces 231
A.3 The Fourier Transform 238
A.4 Problems 240

B Orientation on Numerical Linear Algebra 245
B.1 Direct Methods 245
B.2 Iterative Methods. Relaxation, Overrelaxation,
and Acceleration 246
B.3 Alternating Direction Methods 248
B.4 Preconditioned Conjugate Gradient Methods 249
B.5 Multigrid and Domain Decomposition Methods 250

Bibliography 253

Index 257