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Control Theory in the Plane

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Introduction

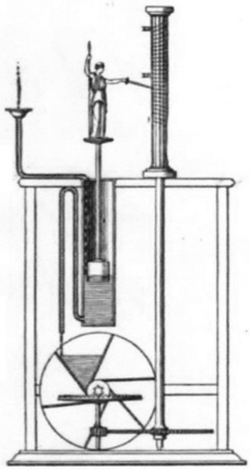
An important scientific innovation rarely makes its way by gradually winning over and converting its opponents … What does happen is that its opponents die out and that the growing generation is familiarised with the idea from the beginning.
(Max Planck, 1936)

Humans have always attempted to influence their environment. Indeed, it seems likely that the understanding of aspects of this environment, and its control, whether by trial-and-error or by actual study and analysis, are crucial to the very process of civilisation.

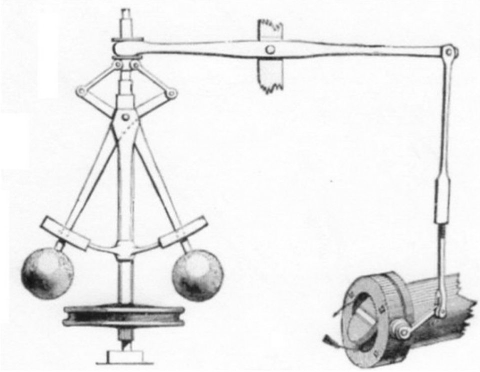
As an illustration, boats and ships were used even in pre-history for fishing, transport, discovery, and trade. Small sailing craft are controlled primarily by working the main-sheet and rudder in conjunction. Once mastered, further experimentation (see e.g. the fifth chapter of the Kon Tiki Expedition, with an entertaining account of the possible use of multiple movable centerboards on a sailing raft) led to a basic change: keeled hulls and corresponding rigging, which made sailing against the wind possible. This was a relatively recent feature: even the far-voyaging Vikings relied primarily on beachable ships and recourse to oars. It was probably crucial in the west-to-east settlement of Oceania, from Taiwan to Easter Island. A 20th century development is the self-steering device, which regulates boat travel automatically under mildly varying wind conditions; but this has had a much smaller social impact.

Devices which might be recognised as automatic regulators appear already in ancient history: clepsydras in the 14th century BC, and water storage overflow controllers in the 3rd BC. A much better case can be made for two inventions from the 18th century. In 1745 Edmund Lee devised the 'automatic fantail' for windmills: here cross-winds engage secondary wind-vanes, which automatically turn the main vanes into the most efficient position, facing the wind (subsequent developments also sensed the wind speed, and automatically feathered the sails: a two-input two-output controller). Some twenty years after constructing his first steam engine in 1765, James Watt invented the centrifugal governor, to hold steady the speed of rotary engines under varying load. The Industrial Revolution, with increasing demand and technological progress, made for the proliferation of such devices.

Japanese deer chaser (shishi odoshi).
Regular clacking as bamboo container fills and empties could have suggested water clocks.
(Author: Shisendo Souzu, Wikimedia Commons)



Water clock (clepsydra).
Entering water raises the figure, pointing to current hour. Spillover turns gears to adjust hour length to season of year (Greek and Roman divided sunrise-to-sunset into twelve hours).
(Source: H.C. Brearly, Time Telling through the Ages)



Centrifugal governor for steam engine.
Higher rotations force metal balls upward, thereby closing off steam pipe.
(Source: Wikimedia Commons)

Possibly the first analyses of control mechanisms appeared in J. Clerk Maxwell's paper On Governors, and Gibbs' On the Equilibrium of Heterogeneous Substances. Jumping ahead, among the highlights of the first half of the 20th century was the development of the thermionic 'valve,' the diode and triode, with their many-faceted and unforeseen applications, leading to present-day transistors; and the successful analysis of their models, the van der Pol equation and the equations of autonomous oscillations. (Here one studied artificially isolated nonlinear systems which seemed to exhibit negative resistance or friction, essentially because energy was fed in from external sources.) On the one hand, engineers realised that differential equations formed a suitable and promising setting for theoretical analysis and design - one aspect of the continuing mathematisation of technology and natural sciences (prompting the title, The Unreasonable Effectiveness of Mathematics in the Natural Sciences, of a 1960 paper by a prominent physicist). On the other, applied mathematicians realised that nonlinear differential equations provided the appropriate, but fiercely difficult, models of control systems; see the proceedings of several influential conferences organised by IFAC, the International Federation for Automatic Control.

The second world war was, also, a period of intense development: radar appeared, then the atomic bomb (an amazing combination of science, engineering, and technology), and computers. The mathematisation process continued, e.g. with the development of operations research in design of efficient policies within complicated situations. Control engineering was involved in connection with power-assisted gun turrets in bombers, in guidance systems for acoustic and magnetic torpedoes, and for the V-weapons. In the last, implementation questions brought in the ‘on-off’ switching devices of regulating and control systems (also called ‘schwarz-weiss’ or ‘bang-bang;’ home thermostats are a common example). These are simple and dependable, economical and dependable; but their effect is far more difficult to analyze than that of the linear or ‘proportional’ elements.

In connection with the automatic guidance of the V2 rocket, Irmgard Flügge-Lotz studied systems incorporating such elements. The simplest is governed by the equation

$$\ddot{x} + \alpha_1 \dot{x} + \alpha_2 x = \beta_1 . \mathrm{sgn}(x + \beta_2 \dot{x}), \tag{1}$$

where the signum function models the on-off switch: in this case an idealised switch, deferring refinements such as hysteresis, time-lag, dead zone (see a series of internal reports, summarised in her post-was book Discontinuous Automatic Control). The question addressed was the choice of the two parameters β_1 and β_2 to achieve desired behaviour, in this case rapid damping of large perturbations. One later formulation, the Letov systems, was a generalisation of (1): in vector notation, in n-space,

$$\dot{x} = Ax + b.\mathrm{sgn}\, \eta(x), \tag{2}$$

where the scalar function $\eta(.)$ is to be chosen appropriately: its selection dictates the design of the feedback device ensuring the required, possibly even optimal, response of the system. (ln (1), $\eta(.)$ is required to be linear, an a priori design limitation; if nonlinear analogue elements are available, this can be relaxed.)

At this stage, in the early 1960s, the emphasis changed dramatically: away from the design of clever devices for automatic regulation to present-day Control theory. Some anonymous engineer or mathematician decided that the formulation in (2) is too complicated, that it conflates two stages which could be, far more simply, tackled consecutively. Thus one could *first* consider

$$\dot{x} = Ax + b.u(t) \tag{3}$$

and seek a function u(.) of time with $-1 \leq u(t) \leq 1$ so as to achieve desired behaviour (open-loop design); *second*, attempt to realise this optimal or suboptimal control u(t) as a function of the current state x,

$$u(t) = \phi(x(t)) \tag{4}$$

(closed-loop or feedback synthesis). If the second stage is unsuccessful, one might augment state space with the needed variables, or even reconsider the problem description entirely. In case the bang-bang principle (at all times utilise maximal capability) applies – and it might not – the function u(.) will have values ± 1 only; then one can express the feedback function $\phi(.)$ in (4) as $\phi(x) = \mathrm{sgn}\,\eta(x)$ for a yet further 'index function' $\eta(.)$ and thereby return to (2).

The idea of the two-step approach (3) – (4) is thus both incisive and versatile; it has worked spectacularly well. It was immediately widely adopted; its author consigned to anonymity, to conform with Stigler's Law ('No scientific discovery is named after its original discoverer:' one wonders whether it was Stigler who discovered this); and the practitioners easily convinced themselves that, of course, this was what they had had in mind all along. In any case, the efforts of a professional historian of technology (or of a PhD candidate) would be most welcome.

One consequence of the open-loop first phase (3) was that the adjective in 'automatic control,' e.g. in the title of IFAC, and of an influential IEEE journal, has become an anachronism: one can consider, separately and subsequently, questions of feedback design (4); but this is not essentially bound together with the problem of controlling, or observing, or optimising the system, as it was in the formulation (2). The horizon of interest expanded immensely.

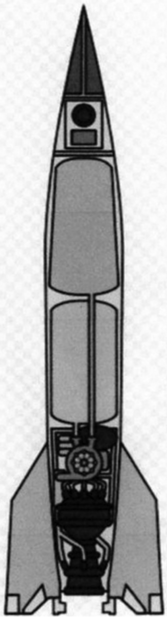
Two points may be made, after this short and idiosyncratic account of the background of control theory.

The first is an attempt at at informal identification: while a major portion of differential equation theory aims at the *analysis* of mathematical models of real systems, control theory is concerned with *synthesis*, the design of systems which are to carry out desired tasks, or behave in a suitable manner. It is in this sense that control theory is more closely associated with engineering than with the natural sciences: the acquisition of useful knowledge rather than knowledge in general.

The second is a sharp disagreement with widely held views and several accounts of the development of control theory. This did not spring into being out of nothing some fifty years ago; nor did it evolve from its slightly disreputable connections with 'cybernetics.' It has had a long history, and has undergone major changes in emphasis and formulation (marked by Pontrjagin's Maximum Principle, and by Bushaw's thesis: but is not co-extensive with these). It does not confine itself to optimisation, even though 'dynamic optimisation' is one of its major topics. It is, however, most fascinating applied mathematics.

(The preceding was the basis for one of the Thursday Kolloquium presentations at the Technische Universität Darmstadt, in spring 1985; and was revised after ensuing comments.)

Dutch-type post windmill, Sønderho, Fanø, Denmark.
The smaller vane at upper right automatically turns the main vane into the prevailing wind direction.
(Author: Chris Nyborg, Wikimedia commons)



V2 rocket (A4 Vergeltungswaffe).
The automatic gyro control (small disk up front) activates jet vanes and air vanes at rear of wings.
(Source: Wikimedia Commons)

We continue with a preface. This book is an introduction to the theory of control systems governed by ordinary differential equations, with special reference to the two-dimensional phase plane. These systems are continuous, deterministic, evolving within finite-dimensional spaces, and with controls subject to a priori bounds.

The first three chapters form the 'general' part of the book; the four subsequent ones are devoted to topics 'in the plane.' Chapter 1 is introductory; it presents some examples that later on are re-visited as the apparatus is developed; and, informally, several concepts. Chapter 2, on systems of ordinary differential equations, contains those portions of the basic theory that are used subsequently; this includes the generalised solutions introduced by Carathéodory. Chapter 3 treats control systems: principal concepts in the general case, and then systems with controls appearing linearly, linear systems, and special bilinear systems.

In differential equation theory, the dynamical systems naturally live in Euclidean n-space R^n; e.g., the two-body problem initially involves a state space of dimension 12: three coordinates of both position and velocity for each of the two bodies. Ingenious trickery then reduces this dimension to 2. In the case of state space dimension $n = 2$, the so-called phase plane, far more can be said than in general. This may be traced to trajectories being planar curves, and the Jordan Curve Theorem: there results the well-known Poincaré-Bendixson theory for planar dynamical systems (cf, e.g., Hájek, Dynamical Systems in the Plane).

Here we exploit the same feature, in the analogous situation of control systems in the plane. Again there appear profound simplifications (Chaps. 4 and 5). Index theory can, with some effort, be carried over and applied (Chap. 6). Green's Theorem,

$$\oint_{\partial D} f dx + g dy = \int\int_{D} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy,$$

whose application in dynamical systems theory seems limited to Bendixson's divergence theorem, becomes a powerful apparatus in optimisation (Chap. 7). Overall, planar control theory appears simpler and more powerful than planar differential equation theory.

The style of exposition is that of a graduate-level text-book. Many chapters begin with expository mini-essays. There is some measure of self-containment (e.g., Chap. 2 on introductory differential equation theory, Sect. 6.1 on classical index theory in the plane); there is an attempt to keep the proofs at a relatively elementary level. Much space is devoted to illustrations and detailed examples; indeed. the manner of exposition might be termed 'example-driven.' Considerable effort has gone into compiling the exercises appended to most sections: the serious student should take these seriously. There is no explicit list of open problems or research and thesis topics; these may be inferred on careful reading. External references are rudimentary; internal ones are as in, 'Eq. (2) of Sect. 3.1.' Innovations in notation or terminology have been avoided, and so has use of technical abbreviations (the lapses are ODE, and QED.)

A portion of the text is loosely based on notes for courses given at Case Western Reserve University, and at Technische Universität Darmstadt. The intended readership is students of applied mathematics who have found attractive the applications of classical differential equations and the calculus of variations: in the relatively recent field of control theory, natural and fundamental conjectures are often easily treated, established or rejected (at least far easier than in, e.g., analytic number theory). The second audience addressed is that of students in engineering (control, systems, electrical, mechanical, aerospace, chemical) who need more background than is provided in the basic mathematics courses, in order to treat analysis, control, and optimisation of systems they already know are important and fascinating. Hopefully, the presented material may also be useful in studying dynamical models in biology and economics.

For the expert who is only browsing through the book, the following lists some possibly unexpected tit-bits. Even for non-linear control systems there are bang-bang theorems (in the planar case: Theorem 7 in Sect. 5.3, Theorem 5 in Sect. 7.1). One can usefully define the index for non-closed and discontinuous paths, see Sect. 6.2. Time optimisation actually can involve (local) maximisation: Example 6 in Sect. 7.2. On a lighter note, there exists only one bilinear system, namely (1) in Sect. 3.6.

Finally, the acknowledgements. I am most grateful for material support, during several stages in the preparation of this book, to: Alexander von Humboldt Stiftung, Case Western Reserve University, Deutsche Forschungsgemeinschaft, Fulbright Program, National Science Foundation, Springer Verlag, TU Darmstadt; most of these were actually multiple instances.

Mrs Carol Larsen typed the manuscript, and suffered without protest the repeated corrections. Most of the figures were prepared on a Macintosh Plus provided bu CWRU, using the author's programs ODE and CODE. Wikimedia Commons was the source for the illustrations in the Introduction. I am grateful to readers who pointed out errors and omissions in the first edition.