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Riemannian Geometry and Geometric Analysis

Fifth Edition

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Dedicated to Shing-Tung Yau,
for so many discussions about
mathematics and Chinese culture

Preface

Riemannian geometry is characterized, and research is oriented towards and shaped by concepts (geodesics, connections, curvature, ...) and objectives, in particular to understand certain classes of (compact) Riemannian manifolds defined by curvature conditions (constant or positive or negative curvature, ...). By way of contrast, geometric analysis is a perhaps somewhat less systematic collection of techniques, for solving extremal problems naturally arising in geometry and for investigating and characterizing their solutions. It turns out that the two fields complement each other very well; geometric analysis offers tools for solving difficult problems in geometry, and Riemannian geometry stimulates progress in geometric analysis by setting ambitious goals.

It is the aim of this book to be a systematic and comprehensive introduction to Riemannian geometry and a representative introduction to the methods of geometric analysis. It attempts a synthesis of geometric and analytic methods in the study of Riemannian manifolds.

The present work is the fifth edition of my textbook on Riemannian geometry and geometric analysis. It has developed on the basis of several graduate courses I taught at the Ruhr-University Bochum and the University of Leipzig. The main new features of the present edition are the systematic inclusion of flow equations and a mathematical treatment of the nonlinear sigma model of quantum field theory. These new topics also led to a systematic reorganization of the other material. Naturally, I have also included several smaller additions and minor corrections (for which I am grateful to several readers).

Let me now briefly describe the contents:

In the first chapter, we introduce the basic geometric concepts, like differentiable manifolds, tangent spaces, vector bundles, vector fields and one-parameter groups of diffeomorphisms, Lie algebras and groups and in particular Riemannian metrics. We also treat the existence of geodesics with two different methods, both of which are quite important in geometric analysis in general. Thus, the reader has the opportunity to understand the basic ideas of those methods in an elementary context before moving on to more difficult versions in subsequent chapters. The first method is based on the local existence and uniqueness of geodesics and will be applied again in Chapter 8 for two-dimensional harmonic maps. The second method is the heat flow method that gained prominence through Perelman's solution of the Poincaré conjecture by the Ricci flow method.

The second chapter introduces de Rham cohomology groups and the essential tools from elliptic PDE for treating these groups. We prove the existence of harmonic forms representing cohomology classes both by a variational method, thereby introducing another of the basic schemes of geometric analysis, and by the heat flow method. The linear setting of cohomology classes allows us to understand some key ideas without the technical difficulties of nonlinear problems.

The third chapter treats the general theory of connections and curvature.

In the fourth chapter, we introduce Jacobi fields, prove the Rauch comparison theorems for Jacobi fields and apply these results to geodesics. We also develop the global geometry of spaces of nonpositive curvature.

These first four chapters treat the more elementary and basic aspects of the subject. Their results will be used in the remaining, more advanced chapters.

The fifth chapter treats Kähler manifolds symmetric spaces as important examples of Riemannian manifolds in detail.

The sixth chapter is devoted to Morse theory and Floer homology.

In the seventh chapter, we treat harmonic maps between Riemannian manifolds. We prove several existence theorems and apply them to Riemannian geometry. The treatment uses an abstract approach based on convexity that should bring out the fundamental structures. We also display a representative sample of techniques from geometric analysis.

In the eighth chapter, we treat harmonic maps from Riemann surfaces. We encounter here the phenomenon of conformal invariance which makes this two-dimensional case distinctively different from the higher dimensional one.

The ninth chapter treats variational problems from quantum field theory, in particular the Ginzburg-Landau, Seiberg-Witten equations, and a mathematical version of the nonlinear supersymmetric sigma model. In mathematical terms, the two-dimensional harmonic map problem is coupled with a Dirac field. The background material on spin geometry and Dirac operators is already developed in earlier chapters. The connections between geometry and physics will be further explored in a forthcoming monograph [144].

A guiding principle for this textbook was that the material in the main body should be self contained. The essential exception is that we use material about Sobolev spaces and linear elliptic an parabolic PDEs without giving proofs. This material is collected in Appendix A. Appendix B collects some elementary topological results about fundamental groups and covering spaces.

Also, in certain places in Chapter 6, we do not present all technical details, but rather explain some points in a more informal manner, in order to keep the size of that chapter within reasonable limits and not to loose the patience of the readers.

We employ both coordinate free intrinsic notations and tensor notations depending on local coordinates. We usually develop a concept in both notations while we sometimes alternate in the proofs. Besides not being a methodological purist, reasons for often prefering the tensor calculus to the more elegant and concise intrinsic one are the following. For the analytic aspects, one often has to employ results about (elliptic) partial differential equations (PDEs), and in order to check that the relevant assumptions like ellipticity hold and in order to make contact with the notations usually employed in PDE theory, one has to write down the differential equation in local coordinates. Also, manifold and important connections have been established between theoretical physics and our subject. In the physical literature, usually the tensor notation is employed, and therefore, familiarity with that notation is necessary for exploring those connections that have been found to be stimulating for the development of mathematics, or promise to be so in the future.

As appendices to most of the paragraphs, we have written sections with the title "Perspectives". The aim of those sections is to place the material in a broader context and explain further results and directions without detailed proofs. The material of these Perspectives will not be used in the main body of the text. Similarly, after Chapter 4, we have inserted a section entitled "A short survey on curvature and topology" that presents an account of many global results of Riemannian geometry not covered in the main text. – At the end of each chapter, some exercises for the reader are given. We assume of the reader sufficient perspicacity to understand our system of numbering and cross-references without further explanations.

The development of the mathematical subject of Geometric Analysis, namely the investigation of analytical questions arising from a geometric context and in turn the application of analytical techniques to geometric problems, is to a large extent due to the work and the influence of Shing-Tung Yau. This book, like its previous editions, is dedicated to him.

I am also grateful to Minjie Chen for dedicated help with the Tex file.

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