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Simplicial Complexes of Graphs

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Preface

This book is a revised version of my 2005 thesis [71] for the degree of Doctor of Philosophy at the Royal Institute of Technology (KTH) in Stockholm. The whole idea of writing a monograph about graph complexes is due to Professor Anders Björner, my scientific advisor. I am deeply grateful for all his comments, remarks, and suggestions during the writing of the thesis and for his very careful reading of the manuscript.

I spent the first years of my academic career at the Department of Mathematics at Stockholm University with Professor Svante Linusson as my advisor. He is the one to get credit for introducing me to the field of graph complexes and also for explaining the fundamentals of discrete Morse theory, the most important tool in this book. Most of the work presented in Chapters 17 and 20 was carried out under the inspiring supervision of Linusson.

The opponent (critical examiner) of my thesis defense was Professor John Shareshian; the examination committee consisted of Professor Boris Shapiro, Professor Richard Stanley, and Professor Michelle Wachs. I am grateful for their valuable feedback that was of great help to me when working on this revision.

The work of transforming the thesis into a book took place at the Technische Universität Berlin and the Massachusetts Institute of Technology. I thank Björner and Professor Günter Ziegler for encouraging me to submit the manuscript to Springer.

Some chapters in this book appear in revised form as journal papers: Chapters 4, 17, and 20 are revised versions of a paper published in the Journal of Combinatorial Theory, Series A [67]. Chapter 5 is a revised version of a paper published in the Electronic Journal of Combinatorics [70]. Chapter 26 is a revised version of a paper published in the SIAM Journal of Discrete Mathematics [72]. I am grateful to several anonymous referees and editors representing these journals, and also to anonymous referees representing the FPSAC conference, who all provided helpful comments and suggestions.

In addition, I thank two anonymous reviewers for this series for providing several useful comments on the manuscript and the editors at Springer for showing patience and being of great help during the preparation of the manuscript.

Finally, and most importantly, I thank family and friends for endless support.

For the reader's convenience, let me list the major revisions compared to the thesis version of 2005:

- Chapter 1 has been extended with a more thorough discussion about applications of graph complexes to problems in other areas of mathematics.
- Recent results about the matching complex M_n and the chessboard complex $\mathsf{M}_{m,n}$ have been incorporated into Sections 11.2.3 and 11.3.2.
- Section 15.4 has been updated with a more precise statement about the Euler characteristic of the complex $\mathsf{DGr}_{n,p}$ of digraphs that are graded modulo p and a shorter proof of a formula for the Euler characteristic of $\mathsf{DGr}_n = \mathsf{DGr}_{n,n+1}$.
- Section 16.3 has been updated with a proof that the complex NXM_n of noncrossing matchings is semi-nonevasive.
- Section 18.5 is new and contains a brief discussion about the complex of disconnected hypergraphs.
- Section 19.4 is new and contains a generalization of the complex NC_n^2 of not 2-connected graphs along with yet another method for computing the homotopy type of NC_n^2. The theory in this section is applied in Section 22.2, which is also new and contains a discussion about the complex DNSC_n^2 of not strongly 2-connected digraphs.
- At the end of Section 23.3, we discuss a recent observation due to Shareshian and Wachs [121] about a connection between the complex NEC_{kp+1}^p of not p-edge-connected graphs on $kp+1$ vertices and the poset $\Pi_{kp+1}^{1 \bmod p}$ of set partitions on $kp+1$ elements in which the size of each part is congruent to 1 modulo p.

Cambridge, MA, *Jakob Jonsson*
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Summary. Let G be a finite graph with vertex set V and edge set E. A *graph complex* on G is an abstract simplicial complex consisting of subsets of E. In particular, we may interpret such a complex as a family of subgraphs of G. The subject of this book is the topology of graph complexes, the emphasis being placed on homology, homotopy type, connectivity degree, Cohen-Macaulayness, and Euler characteristic.

We are particularly interested in the case that G is the complete graph on V. *Monotone graph properties* are complexes on such a graph satisfying the additional condition that they are invariant under permutations of V. Some well-studied monotone graph properties that we discuss in this book are complexes of matchings, forests, bipartite graphs, disconnected graphs, and not 2-connected graphs. We present new results about several other monotone graph properties, including complexes of not 3-connected graphs and graphs not coverable by p vertices.

Imagining the vertices as the corners of a regular polygon, we obtain another important class consisting of those graph complexes that are invariant under the natural action of the dihedral group on this polygon. The most famous example is the associahedron, whose faces are graphs without crossings inside the polygon. Restricting to matchings, forests, or bipartite graphs, we obtain other interesting complexes of noncrossing graphs. We also examine a certain "dihedral" variant of connectivity.

The third class to be examined is the class of digraph complexes. Some well-studied examples are complexes of acyclic digraphs and not strongly connected digraphs. We present new results about a few other digraph complexes, including complexes of graded digraphs and non-spanning digraphs.

Many of our proofs are based on Robin Forman's discrete version of Morse theory. As a byproduct, this book provides a loosely defined toolbox for attacking problems in topological combinatorics via discrete Morse theory. In terms of simplicity and power, arguably the most efficient tool is Forman's divide and conquer approach via decision trees, which we successfully apply to a large number of graph and digraph complexes.

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Part I Introduction and Basic Concepts