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Weighted Littlewood-Paley Theory and Exponential-Square Integrability

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I dedicate this book to my parents, James and Joyce Wilson.

Preface

Littlewood-Paley theory can be thought of as a profound generalization of the Pythagorean theorem. If $x \in \mathbf{R}^d$—say, $x = (x_1, x_2, \ldots, x_d)$—then we define x's norm, $\|x\|$, to be $(\sum_1^d x_n^2)^{1/2}$. This norm has the good property that, if $y = (y_1, y_2, \ldots, y_d)$ is any other vector in $\mathbf{R}^d$, and $|y_n| \leq |x_n|$ for each n, then $\|y\| \leq \|x\|$. In other words, the size of x, as measured by the norm function, is determined entirely by the sizes of x's components. This remains true if we let the dimension d increase to infinity, and define the norm of a vector (actually, an infinite sequence) $x = (x_1, x_2, \ldots)$ to be $\|x\| \equiv (\sum_1^\infty x_n^2)^{1/2}$.

In analysis it is often convenient (and indispensable) to decompose functions f into infinite series,

$$f(x) = \sum \lambda_n \phi_n(x), \tag{0.1}$$

where the functions ϕ_n come from some standard family (such as the Fourier system) and the λ_n's are complex numbers. (For the time being we will not specify how the series 0.1 is supposed to converge.) Typically the coefficients λ_n are defined by integrals of f against some other functions ψ_n. If we are interested about convergence in the sense of L^2 (or "mean-square"), and if the ϕ_n's comprise a *complete orthonormal family*, then each ψ_n can be taken to be $\bar{\phi}_n$, the complex conjugate of ϕ_n; i.e.,

$$\lambda_n = \int f(x)\, \bar{\phi}_n(x)\, dx,$$

and we have

$$\int |f(x)|^2\, dx = \sum |\lambda_n|^2.$$

(For the time being we will not specify the *domain* on which f and the ϕ_n's are defined.) If we are only interested in L^2 functions, then the natural norm,

$$\|f\|_2 \equiv \left(\int |f(x)|^2\, dx\right)^{1/2} = \left(\sum |\lambda_n|^2\right)^{1/2},$$

has the same domination property possessed by the Euclidean norm on $\mathbf{R}^d$: if $g = \sum \gamma_n \phi_n$ and $|\gamma_n| \le |\lambda_n|$ for all n, then $\|g\|_2 \le \|f\|_2$. Even better, if, for some $\epsilon > 0$, we have $|\gamma_n| \le \epsilon|\lambda_n|$ for all n, then $\|g\|_2 \le \epsilon\|f\|_2$.

Unfortunately, L^2 is not always the most useful function space for a given problem. We might want to work in L^4, with its norm defined by

$$\|f\|_4 \equiv \left(\int |f(x)|^4 \, dx \right)^{1/4}.$$

To make things specific, let's suppose that our functions are defined on $[0, 1)$. The collection $\{\exp(2\pi inx)\}_{-\infty}^{\infty}$ defines a complete orthonormal family in $L^2[0, 1)$. Now, if $f \in L^4[0, 1)$, then the coefficients

$$\lambda_n \equiv \int_0^1 f(x) \exp(-2\pi inx) \, dx$$

are defined, and the infinite series,

$$\sum_{-\infty}^{\infty} \lambda_n \exp(2\pi inx),$$

converges to f in the L^4 sense, if we sum it up right. But the domination property fails in a very strong sense. Given $f \in L^4$, and given an integrable function g such $|\gamma_n| \le |\lambda_n|$ for all n, where

$$\gamma_n = \int_0^1 g(x) \exp(-2\pi inx) \, dx,$$

there is no reason to expect that $\|g\|_4$ is even finite, let alone controlled by $\|f\|_4$.

Littlewood-Paley theory provides a way to *almost* preserve the domination property. To each function f, one associates something called the *square function of* f, denoted $S(f)$. (Actually, the square function comes in many guises, but we will not go into that now.) Each square function is defined via inner products with a fixed collection of functions. Sometimes this collection is a complete orthonormal family for L^2, but it doesn't have to be. The square function $S(f)(x)$ is defined as a weighted sum (or integral) of the squares of the inner products, $|\langle f, \phi\rangle|^2$, where ϕ belongs to the fixed collection. The function $S(f)(x)$ varies from point to point, but, if f and g are two functions such that $|\langle g, \phi\rangle| \le |\langle f, \phi\rangle|$ for all ϕ, then $S(g)(x) \le S(f)(x)$ everywhere. The square function $S(f)$ also has the property that, if $1 < p < \infty$, and $f \in L^p$, the L^p norms of $S(f)$ and f are *comparable.*

The combination of these two facts—domination plus comparablility—lets us, in many situations, reduce the analysis of infinite series of functions,

$$f(x) = \sum \lambda_i \phi_i(x),$$

to the analysis of infinite series of *non-negative* functions,

$$S(f)(x) = \left(\sum |\gamma_i|^2 |\psi_i(x)|^2\right)^{1/2};$$

and that greatly simplifies things. We have already mentioned the practice, common in analysis, of cutting a function into infinitely many pieces. Typically we do this to solve a problem, such as a PDE. We break the data into infinitely many pieces, solve the problem on each piece, and then sum the "piece-wise" solutions. The sums encountered this way are likely to contain a lot of complicated cancelations. Littlewood-Paley theory lets us control them by means of sums that have *no* cancelations.

The mutual control between $|f|$ and $S(f)$ is very tight. We will soon show that, if f is a bounded function defined on $[0,1)$, there is a positive α such that $\exp(\alpha(S(f))^2)$ is integrable on $[0,1)$—*and vice versa.* (This is not quite like saying that $|f|$ and $S(f)$ are pointwise comparable, but in many applications they might as well be.) This tight control is expressed quantitatively in terms of weighted norm inequalities. The reader will learn some sufficient (and not terribly restrictive) conditions on pairs of weights which ensure that

$$\int |f(x)|^p \, v \, dx \leq \int (S(f)(x))^p \, w \, dx \tag{0.2}$$

or

$$\int (S(f)(x))^p \, v \, dx \leq \int |f(x)|^p \, w \, dx \tag{0.3}$$

holds for all f in suitable test classes, for various ranges of p (usually, $1 < p < \infty$). He will also learn some necessary conditions for such inequalities.

The usefulness of the square function (in its many guises) comes chiefly from the fact that, for many linear operators T, $S(T(f))$, the square function of $T(f)$, is bounded pointwise by a function $\tilde{S}(f)$, where $\tilde{S}(\cdot)$ is an operator similar to—and satisfying estimates similar to—$S(\cdot)$. This makes it possible to understand the behavior of T, because one can say: $|T(f)|$ is controlled by $S(T(f))$, which is controlled by $\tilde{S}(f)$, which is controlled by $|f|$. Obviously, the closer the connection between $|f|$ and $S(f)$, the more efficient this process will be. The exponential-square results (and the corresponding weighted norm inequalities) imply that this connection is pretty close.

We have tried to make this book self-contained, not too long, and accessible to non-experts. We have also tried to avoid excessive overlap with other books on weighted norm inequalities. Therefore we have not treated every topic of relevance to weighted Littlewood-Paley theory. We have not touched on multi-parameter analysis at all, and we have dealt only briefly with vector-valued inequalities. We discuss A_p weights mainly with reference to the square function and singular integral operators. We prove the boundedness of the Hardy-Littlewood operator on $L^p(w)$ for $w \in A_p$ and we prove an extrapolation result—because we need both—but we don't prove A_p factorization or the Rubio de Francia extrapolation theorem, excellent treatments of which can be found in many books (e.g., [16] and [24]).

The book is laid out this way. Chapter 1 covers some basic facts from harmonic analysis. Most of the material there will be review for many people, but we have tried to present it so as not to intimidate the non-experts. Chapter 2 introduces the one-dimensional dyadic square function and proves some of its properties; it also introduces a few more techniques from harmonic analysis. In chapter 3 we prove the exponential-square estimates mentioned above (in one dimension only). These lead to an in-depth look at weighted norm inequalities. In chapter 4 we extend the results of the preceding chapters to d dimensions and to continuous analogues of the dyadic square function.

Chapters 5, 6, and 7 are devoted to the Calderón reproducing formula. The Calderón formula provides a canonical way of expressing "arbitrary" functions as linear sums of special, smooth, compactly supported functions. It is the foundation of wavelet theory. Aside from some casual remarks[1], we don't talk about wavelets. The expert will see the close connections between wavelets and the material in chapters 5–7. The non-expert doesn't have to worry about them to understand the material; but, should he ever encounter wavelets, a good grasp of the Calderón formula will come in very handy. We have devoted three chapters to it because we believe the reader will gain more by seeing essentially the same problem (the convergence of the Calderón integral formula) treated in increasing levels of generality, than in having one big portmanteau theorem dumped onto his lap. The portmanteau theorem (Theorem 7.1) does come; but we trust that, when it does, the reader is more than able to bear its weight.

Chapters 8 and 9 give some straightforward applications of weighted Littlewood-Paley theory to the analysis of Schrödinger and singular integral operators. This material could easily have come after that in chapter 10, but we felt that, where it is, it gave the reader a well-earned break from purely theoretical discussions.

In chapter 10 we return to theory. The scale of Orlicz spaces (which includes that of L^p spaces for $1 \leq p \leq \infty$) provides a flexible way of keeping track of the integrability properties of functions. It is very useful in the study of weighted norm inequalities. The material here *could* have come at the very beginning, but we felt that the reader would understand this theory better if he first saw the need for it.

As an application of Orlicz space theory, chapter 11 presents a different proof of Theorem 3.8 from chapter 3. This ingenious argument, due to Fedor Nazarov, completely avoids the use of good-λ inequalities (which we introduce in chapter 2). These have been a mainstay of analysis since the early 1970s. In the opinion of some researchers, they have also become a crutch. We are neutral on this issue, but please see our note at the end of chapter 2.

Chapter 12 applies the theory from the preceding chapters to give a new (and, we hope, accessible) proof of the Hörmander-Mihlin multiplier theorem. Chapter 13 extends the main weighted norm results from earlier chapters

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[1] Like this one.

to the ℓ^2-valued setting. In chapter 14 we prove one theorem (Khinchin's Inequalities), but our discussion there is mainly philosophical. We look at what Littlewood-Paley theory can tell us about *pointwise* summation errors of Haar function expansions.

We have put exercises at the end of almost every chapter. Some of them expand on topics treated in the text; some tie up loose ends in proofs; some are referred to later in the book. We encourage the reader to understand all of them and to attempt at least a few. (We have supplied hints for the more difficult ones.)

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