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Evolution Algebras and their Applications

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To my parents

Bi-Yuan Tian and Yu-Mei Liu

My father, the only person I know who can operate two abaci using his left and right hand simultaneously in his business.

Preface

In this book, we introduce a new type of algebra, which we call evolution algebras. These are algebras in which the multiplication tables are of a special type. They are motivated by evolution laws of genetics. We view alleles (or organelles or cells, etc.) as generators of algebras. Therefore we define the multiplication of two “alleles” G_i and G_j by $G_i \cdot G_j = 0$ if $i \neq j$. However, $G_i \cdot G_i$ is viewed as “self-reproduction,” so that $G_i \cdot G_i = \sum_j p_{ij} G_j$, where the summation is taken over all generators G_j . Thus, reproduction in genetics is represented by multiplication in algebra. It seems obvious that this type of algebra is nonassociative, but commutative. When the p_{ij} s form Markovian transition probabilities, the properties of algebras are associated with properties of Markov chains. Markov chains allow us to develop an algebra theory at deeper hierarchical levels than standard algebras. After we introduce several new algebraic concepts, particularly algebraic persistency, algebraic transiency, algebraic periodicity, and their relative versions, we establish hierarchical structures for evolution algebras in Chapter 3. The analysis developed in this book, particularly in Chapter 4, enables us to take a new perspective on Markov process theory and to derive new algebraic properties for Markov chains at the same time. We see that any Markov chain has a dynamical hierarchy and a probabilistic flow that is moving with invariance through this hierarchy. We also see that Markov chains can be classified by the skeleton-shape classification of their evolution algebras. Remarkably, when applied to non-Mendelian genetics, particularly organelle heredity, evolution algebras can explain establishment of homoplasmy from heteroplasmic cell population and the coexistence of mitochondrial triplasmy, and can also predict all possible mechanisms to establish the homoplasmy of cell population. Actually, these mechanisms are hypothetical mechanisms in current mitochondrial disease research. By using evolution algebras, it is easy to identify different genetic patterns from the complexity of the progenies of *Phytophthora infestans* that cause the late blight of potatoes and tomatoes. Evolution algebras have many connections with other fields of mathematics, such as graph theory, group theory, knot theory, 3-manifolds, and Ihara-Selberg zeta functions. Evolution

algebras provide a theoretical framework to unify many phenomena. Among the further research topics related to evolution algebras and other fields, the most significant topic perhaps is to develop a continuous evolution algebra theory for continuous time dynamical systems.

The intended audience of this book includes graduate students and researchers with interest in theoretical biology, genetics, Markov processes, graph theory, and nonassociative algebras and their applications.

Professor Jean-Michel Morel gave me a lot of support and encouragement, which enabled me to take the step to publish my research results as a book. Other editors and staff in LNM made efforts to find reviewers and edit my book. Here, I wish to express my great thanks to them.

I thank Professor Michael T. Clegg for his stimulating problems in coalescent theory. From that point, I began to study genetics and stochastic processes. I am greatly indebted to Professor Xiao-Song Lin, my Ph.D advisor, for his valuable advice and long-time guidance. I am thankful to professors Bai-Lian Larry Li, Michel L. Lapidus, and Barry Arnold for their valuable suggestions. It gives me great pleasure to thank Professors Bun Wong, Yat Sun Poon, Shizhong Xu, Keh-Shin Lii, Peter March, Dennis Pearl, Raymond L. Orbach, Murray Bremner, Yuan Lou, and Yang Kuang for their encouragement. I also thank Professor C. William Birky Jr. for his explanation of non-Mendelian genetics through e-mails. I acknowledge Professor Winfried Just for his suggestions of writing style of the book and a formula in Chapter 3. I am grateful to my current mentor, Professor Avner Friedman, for his detailed and cherished suggestions on the research in this book and my other research directions. I thank three reviewers for their suggestions and constructive comments.

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April, 2007

Contents

1	Introduction	1
2	Motivations	9
2.1	Examples from Biology	9
2.1.1	Asexual propagation	9
2.1.2	Gametic algebras in asexual inheritance	10
2.1.3	The Wright-Fisher model	11
2.2	Examples from Physics	12
2.2.1	Particles moving in a discrete space	12
2.2.2	Flows in a discrete space (networks)	12
2.2.3	Feynman graphs	13
2.3	Examples from Topology	15
2.3.1	Motions of particles in a 3-manifold	15
2.3.2	Random walks on braids with negative probabilities ...	15
2.4	Examples from Probability Theory	16
2.4.1	Stochastic processes	16
3	Evolution Algebras	17
3.1	Definitions and Basic Properties	17
3.1.1	Departure point	17
3.1.2	Existence of unity elements	22
3.1.3	Basic definitions	23
3.1.4	Ideals of an evolution algebra	24
3.1.5	Quotients of an evolution algebra	25
3.1.6	Occurrence relations	26
3.1.7	Several interesting identities	27
3.2	Evolution Operators and Multiplication Algebras	28
3.2.1	Evolution operators	29
3.2.2	Changes of generator sets (Transformations of natural bases)	30
3.2.3	“Rigidity” of generator sets of an evolution algebra ...	31

3.2.4	The automorphism group of an evolution algebra	32
3.2.5	The multiplication algebra of an evolution algebra	33
3.2.6	The derived Lie algebra of an evolution algebra	34
3.2.7	The centroid of an evolution algebra	35
3.3	Nonassociative Banach Algebras	36
3.3.1	Definition of a norm over an evolution algebra	37
3.3.2	An evolution algebra as a Banach space	38
3.4	Periodicity and Algebraic Persistency	39
3.4.1	Periodicity of a generator in an evolution algebra	39
3.4.2	Algebraic persistency and algebraic transiency	42
3.5	Hierarchy of an Evolution Algebra	43
3.5.1	Periodicity of a simple evolution algebra	44
3.5.2	Semidirect-sum decomposition of an evolution algebra	45
3.5.3	Hierarchy of an evolution algebra	46
3.5.4	Reducibility of an evolution algebra	49
4	Evolution Algebras and Markov Chains	53
4.1	A Markov Chain and Its Evolution Algebra	53
4.1.1	Markov chains (discrete time)	53
4.1.2	The evolution algebra determined by a Markov chain	54
4.1.3	The Chapman–Kolmogorov equation	56
4.1.4	Concepts related to evolution operators	58
4.1.5	Basic algebraic properties of Markov chains	58
4.2	Algebraic Persistency and Probabilistic Persistency	60
4.2.1	Destination operator of evolution algebra M_X	60
4.2.2	On the loss of coefficients (probabilities)	64
4.2.3	On the conservation of coefficients (probabilities)	67
4.2.4	Certain interpretations	68
4.2.5	Algebraic periodicity and probabilistic periodicity	69
4.3	Spectrum Theory of Evolution Algebras	69
4.3.1	Invariance of a probability flow	69
4.3.2	Spectrum of a simple evolution algebra	70
4.3.3	Spectrum of an evolution algebra at zeroth level	75
4.4	Hierarchies of General Markov Chains and Beyond	76
4.4.1	Hierarchy of a general Markov chain	76
4.4.2	Structure at the 0th level in a hierarchy	77
4.4.3	1st structure of a hierarchy	80
4.4.4	k th structure of a hierarchy	81
4.4.5	Regular evolution algebras	83
4.4.6	Reduced structure of evolution algebra M_X	86
4.4.7	Examples and applications	87

5 Evolution Algebras and Non-Mendelian Genetics 91

5.1 History of General Genetic Algebras 91

5.2 Non-Mendelian Genetics and Its Algebraic Formulation 93

5.2.1 Some terms in population genetics 93

5.2.2 Mendelian vs. non-Mendelian genetics 94

5.2.3 Algebraic formulation of non-Mendelian genetics 95

5.3 Algebras of Organelle Population Genetics 96

5.3.1 Heteroplasmy and homoplasmy 96

5.3.2 Coexistence of triplasmmy 98

5.4 Algebraic Structures of Asexual Progenies of *Phytophthora infestans* 100

5.4.1 Basic biology of *Phytophthora infestans* 101

5.4.2 Algebras of progenies of *Phytophthora infestans* 102

6 Further Results and Research Topics 109

6.1 Beginning of Evolution Algebras and Graph Theory 109

6.2 Further Research Topics 113

6.2.1 Evolution algebras and graph theory 113

6.2.2 Evolution algebras and group theory, knot theory 114

6.2.3 Evolution algebras and Ihara-Selberg zeta function 115

6.2.4 Continuous evolution algebras 115

6.2.5 Algebraic statistical physics models and applications 115

6.2.6 Evolution algebras and 3-manifolds 116

6.2.7 Evolution algebras and phylogenetic trees, coalescent theory 116

6.3 Background Literature 116

References 119

Index 123