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Financial Markets in Continuous Time

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Preface

In modern financial practice, asset prices are modelled by means of stochastic processes. Continuous-time stochastic calculus thus plays a central role in financial modelling. The approach has its roots in the foundational work of Black, Scholes and Merton. Asset prices are further assumed to be rationalizable, that is, determined by the equality of supply and demand in some market. This approach has its roots in the work of Arrow, Debreu and McKenzie on general equilibrium.

This book is aimed at graduate students in mathematics or finance. Its objective is to develop in continuous time the valuation of asset prices and the theory of the equilibrium of financial markets in the complete market case (the theory of optimal portfolio and consumption choice being considered as part of equilibrium theory).

Firstly, various models with a finite number of states and dates are reviewed, in order to make the book accessible to masters students and to provide the economic foundations of the subject.

Four chapters are then concerned with the valuation of asset prices: one chapter is devoted to the Black–Scholes formula and its extensions, another to the yield curve and the valuation of interest rate products, another to the problems linked to market incompleteness, and a final chapter covers exotic options.

Three chapters deal with “equilibrium theory”. One chapter studies the problem of the optimal choice of portfolio and consumption for a representative agent in the complete market case. Another brings together a number of results from the theory of general equilibrium and the theory of equilibrium in financial markets, in a discrete framework. A third chapter deals with the

Radner equilibrium in continuous time in the complete market case, and its financial applications.

Appendices provide a basic presentation of Brownian motion and of numerical solutions to partial differential equations.

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ROSE-ANNE DANA
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