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Jie Shen · Tao Tang · Li-Lian Wang

Spectral Methods

Algorithms, Analysis and Applications

Springer

Jie Shen
Department of Mathematics
Purdue University
N. University St. 150
West Lafayette, IN 47907-2067
USA
shen@math.purdue.edu

Tao Tang
Department of Mathematics
Hong Kong Baptist University
Waterloo Road 224
Kowloon
Hong Kong SAR
ttang@hkbu.edu.hk

Li-Lian Wang
Division of Mathematical Sciences
School of Physical & Mathematical Sciences
Nanyang Technological University
21 Nanyang Link
637371
Singapore
lilian@ntu.edu.sg

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Preface

This book is developed from lecture notes of graduate courses taught over the years by the authors at the Pennsylvania State University, Purdue University, Hong Kong Baptist University and Nanyang Technological University of Singapore.

The aim of the book is to provide

- A detailed presentation of basic spectral algorithms
- A systematical presentation of basic convergence theory and error analysis for spectral methods
- Some illustrative applications of spectral methods

For many basic algorithms presented in the book, we provide Matlab codes (which will be made available online) which contain additional programming details beyond the mathematical formulas, so that the readers can easily use or modify these codes to suite their need. We believe that these Matlab codes will help the readers to have a better understanding of these spectral algorithms and provide a useful starting point for developing their own application codes.

There are already quite a few monographs/books on spectral methods. The classical books by Gottlieb and Orszag (1977) and by Canuto et al. (1987)[1] were intended for researchers and advanced graduate students, and they are excellent references for the historical aspects of spectral methods as well as in depth presentations of various techniques and applications in computational fluid dynamics. The book by Boyd (2001) focused on the Fourier and Chebyshev methods with emphasis on implementations and applications. The book by Trefethen (2000) gave an excellent exposition on the spectral-collocation methods through a set of elegant Matlab routines. The books by Deville et al. (2002) and by Karniadakis and Sherwin (2005) concentrated on the spectral-element methods with details on parallel implementations and applications in fluid dynamics, while the more recent book by Hesthaven and Warburton (2008) focused on the discontinuous Galerkin methods with a nodal spectral-element approach. On the other hand, Hesthaven et al. (2007) focused on

[1] An updated and expanded version of Canuto et al. (1987) is recently published. This new version Canuto et al. (2006, 2007) incorporated many new developments made in the last 20 years and provided a more systematical treatment for spectral methods.

the spectral methods for time-dependent problems with a particular emphasis on hyperbolic equations and problems with non-smooth solutions. The book length article by Bernardi and Maday (1997) and their monograph in French Bernardi and Maday (1992a) provided an excellent exposition on the basic approximation theory of spectral methods with a particular emphasis on Stokes equations, while the monograph (Shen and Tang 2006) presented a basic introduction in a lecture note style to the implementation and analysis of spectral methods. The emphasis of the book by Guo (1998b), on the other hand, was on numerical analysis of spectral methods for nonlinear evolution problems. Finally, spectral methods have been playing a very significant role in dealing with stochastic differential equations and uncertainty quantifications, and we refer to the recent books by Le Maître and Knio (2010) and by Xiu (2010) on these emerging topics.

The current book attempts to provide a self-contained presentation for the construction, implementation and analysis of efficient spectral algorithms for some model equations, of elliptic, dispersive and parabolic type, which have wide applications in science and engineering. It strives to provide a systematical approach based on variational formulations for both algorithm development and numerical analysis. Some of the unique features of the current book are

- Our analysis is based on the non-uniformly weighted Sobolev spaces which lead to simplified analysis and more precise estimates, particularly for problems with corner singularities. We also advocate the use of the generalized Jacobi polynomials which are particularly useful for dealing with boundary value problems.
- We develop efficient spectral algorithms and present their error analysis for Volterra integral equations, higher-order differential equations, problems in unbounded domains and in high-dimensional domains. These topics have rarely been covered in detail in the existing books on spectral methods.
- We provide online a set of well structured Matlab codes which can be easily modified and expanded or rewritten in other programming languages.

The Matlab codes as well as corrections/updates to the book will be available at http://www.math.purdue.edu/~shen/STWbook. In case this site becomes unavailable due to unforeseen circumstances in the future, the readers are advised to check the Springer Web site for the updated Web link on the book.

We do not attempt to provide in this book an exhaustive account on the wide range of topics that spectral methods have had impact on. In particular, we do not include some important topics such as spectral methods for hyperbolic equations and spectral-element methods, partly because these topics do not fit well in our uniform framework, and mostly because there are already some excellent books mentioned above on these topics. As such, no attempt is made to provide a comprehensive list of references on the spectral methods. The cited references reflect the topics covered in the book, but inevitably, the authors' bias. While we strive for correctness, it is most likely that errors still exist. We welcome comments, suggestions and corrections.

The book can be used as a textbook for graduate students in both mathematics and other science/engineering. Mathematical analysis and applications are organized mostly at the end of each chapter and presented in such a way that they can be skipped without affecting the understanding of algorithms in the following chapters. The first four chapters and Sects. 8.1–8.4 provide the basic ingredients on Fourier and polynomial approximations and essential strategies for developing efficient spectral-Galerkin and spectral-collocation algorithms. Section 8.5 deals with sparse spectral methods for high-dimensional problems. The topics in Chaps. 5, 6 and 7 are independent of each other so the readers can choose according to their need. Applications covered in Chap. 9, except for a slight dependence on Sects. 9.4–9.5, are also independent of each other. For the readers' convenience, we provide in the Appendices some essential mathematical concepts, basic iterative algorithms and commonly used time discretization schemes.

The book is also intended as a reference for active practitioners and researchers of spectral methods. The prerequisite for the book includes standard entry-level graduate courses in Numerical Analysis, Functional Analysis and Partial Differential Equations (PDEs). Some knowledge on numerical approximations of PDEs will be helpful in understanding the convergence theory and error analysis but hardly necessary for understanding the numerical algorithms presented in this book.

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Indiana, USA — Jie Shen
Hong Kong, China — Tao Tang
Singapore — Li-Lian Wang

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Symbol List

Common Notation

$\mathbb{C}$	Set of all complex numbers
$\mathbb{R}$	Set of all real numbers
$\mathbb{Z}$	Set of all integers
$\mathbb{N}$	Set of all nonnegative integers
P_N	Set of all real polynomials of degree $\leq N$
i	Complex unit, i.e., $\mathrm{i} = \sqrt{-1}$
δ_{mn}	Kronecker Delta symbol
Γ	Gamma function defined in (A.1)
$\cong$	$z_n \cong w_n$ means that for $w_n \neq 0$, $z_n/w_n \to 1$ as $n \to \infty$
$\sim$	$z_n \sim w_n$ means that for $w_n \neq 0$, $z_n/w_n \to C$ (independent of n) as $n \to \infty$
$\lesssim$	$z_n \lesssim w_n$ means that $z_n \leq C w_n$ with C independent of n

Orthogonal Polynomials/Functions

L_n	Legendre polynomial of degree n defined in (3.168)
T_n	Chebyshev polynomial of degree n defined in (3.207)
$J_n^{\alpha,\beta}$	Jacobi polynomial of degree n with parameter (α,β) defined in (3.110)
$J_n^{k,l}$	generalized Jacobi polynomial of degree n with $k,l \in \mathbb{Z}$ defined in (6.1)
$\mathscr{L}_n$	Laguerre polynomial of degree n defined in (7.4) with $\alpha = 0$
$\widehat{\mathscr{L}}_n$	Laguerre function of degree n defined in (7.16) with $\alpha = 0$
$\mathscr{L}_n^{(\alpha)}$	generalized Laguerre polynomial of degree n with parameter α defined in (7.4)
$\widehat{\mathscr{L}}_n^{(\alpha)}$	generalized Laguerre function of degree n with parameter α defined in (7.16)
H_n	Hermite polynomial of degree n defined in (7.58)
$\widehat{H}_n$	Hermite function of degree n defined in (7.71)

Weight Functions and Weighted Spaces of Functions

ω	A generic non-negative weight function
$\omega^{\alpha,\beta}$	Jacobi weight function: $\omega^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$
ω_α	Weight function associated with $\mathscr{L}_n^{(\alpha)}$, i.e., $\omega_\alpha(x) = x^\alpha e^{-x}$
$\hat{\omega}_\alpha$	Weight function associated with $\widehat{\mathscr{L}}_n^{(\alpha)}$, i.e., $\hat{\omega}_\alpha(x) = x^\alpha$
$L^p(\Omega)$	L^p-space on Ω with $1 \le p \le \infty$
$H^r(\Omega)$	Sobolev space on Ω
$H^r_\omega(\Omega)$	Weighted Sobolev space on Ω
$B^r_{\alpha,\beta}(I^d)$	Non-uniformly Jacobi-weighted Sobolev space defined in (3.251) ($d=1$) and in (8.125) with vector-valued $\boldsymbol{\alpha}, \boldsymbol{\beta}$
$B^r_\alpha(\mathbb{R}_+)$	Non-uniformly weighted Sobolev space defined in (7.103)
$\hat{B}^r_\alpha(\mathbb{R}_+)$	Non-uniformly weighted Sobolev space defined in (7.110)
$\mathbb{K}^r_{\boldsymbol{\alpha},\boldsymbol{\beta}}(I^d)$	Jacobi-weighted Korobov-type space defined in (8.190)

Inner Products and Norms

$(\cdot,\cdot)_\omega$	Inner product of $L^2_\omega(\Omega)$
$(\cdot,\cdot)$	Inner product of $L^2(\Omega)$
$\|\cdot\|_\omega$	Norm of $L^2_\omega(\Omega)$
$\|\cdot\|_{r,\omega}$	Norm of $H^r_\omega(\Omega)$
$|\cdot|_{r,\omega}$	Semi-norm of $H^r_\omega(\Omega)$
$\|\cdot\|$	Norm of $L^2(\Omega)$
$\|\cdot\|_r$	Norm of $H^r(\Omega)$
$|\cdot|_r$	Semi-norm of $H^r(\Omega)$
$\|\cdot\|_\infty$	Norm of $L^\infty(\Omega)$
$\langle\cdot,\cdot\rangle_{N,\omega}$	Discrete inner product associated with a Gauss-type quadrature
$\langle\cdot,\cdot\rangle_N$	$\langle\cdot,\cdot\rangle_N = \langle\cdot,\cdot\rangle_{N,\omega}$ with $\omega \equiv 1$
$\|\cdot\|_{N,\omega}$	Discrete norm associated with $\langle\cdot,\cdot\rangle_{N,\omega}$

One-Dimensional Projection/Interpolation Operators

$\pi_N^{\alpha,\beta}$	$L^2_{\omega^{\alpha,\beta}}$-orthogonal projection operator defined in (3.249)
$\pi^1_{N,\alpha,\beta}$	$H^1_{\omega^{\alpha,\beta}}$-orthogonal projection operator defined in (3.269)
$\pi^{1,0}_{N,\alpha,\beta}$	$H^1_{0,\omega^{\alpha,\beta}}$-orthogonal projection operator defined in (3.290)
$I_N^{\alpha,\beta}$	Jacobi-Gauss-type interpolation operator
$\pi_N,\ I_N$	Operators $\pi_N^{\alpha,\beta},\ I_N^{\alpha,\beta}$ with $\alpha=\beta=0$
$\pi_N^c,\ I_N^c$	Operators $\pi_N^{\alpha,\beta}, I_N^{\alpha,\beta}$ with $\alpha=\beta=-1/2$
$\Pi_{N,\alpha}$	Orthogonal projection operator in $L^2_{\omega_\alpha}(\mathbb{R}_+)$ defined in (7.102)
$\hat{\Pi}_{N,\alpha}$	Orthogonal projection operator in $L^2_{\hat{\omega}_\alpha}(\mathbb{R}_+)$ defined in (7.109)
Π_N	Orthogonal projection operator in $L^2_\omega(\mathbb{R})$ with $\omega = e^{-x^2}$ defined in (7.125)
$\hat{\Pi}_N$	Orthogonal projection operator defined in (7.128)
$I_N^\alpha,\ \hat{I}_N^\alpha$	Laguerre-Gauss-type interpolation operators
$I_N^h,\ \hat{I}_N^h$	Hermite-Gauss interpolation operators