Lecture Notes in Mathematics 1958

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Compactifying Moduli Spaces for Abelian Varieties

Springer

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ISBN: 978-3-540-70518-5 e-ISBN: 978-3-540-70519-2
DOI: 10.1007/978-3-540-70519-2

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2008931163

Mathematics Subject Classification (2000): 14K10, 14K25

Cover design: SPi Publishing Services

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

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Summary

The problem of compactifying the moduli space $\mathscr{A}_g$ of principally polarized abelian varieties has a long and rich history. The majority of recent work has focused on the toroidal compactifications constructed over $\mathbb{C}$ by Mumford and his coworkers, and over $\mathbb{Z}$ by Chai and Faltings. The main drawback of these compactifications is that they are not canonical and do not represent any reasonable moduli problem on the category of schemes. The starting point for this work is the realization of Alexeev and Nakamura that there is a canonical compactification of the moduli space of principally polarized abelian varieties. Indeed Alexeev describes a moduli problem representable by a proper algebraic stack over $\mathbb{Z}$ which contains $\mathscr{A}_g$ as a dense open subset of one of its irreducible components.

In this text we explain how, using logarithmic structures in the sense of Fontaine, Illusie, and Kato, one can define a moduli problem "carving out" the main component in Alexeev's space. We also explain how to generalize the theory to higher degree polarizations and discuss various applications to moduli spaces for abelian varieties with level structure.

If d and g are positive integers we construct a proper algebraic stack with finite diagonal $\overline{\mathscr{A}}_{g,d}$ over $\mathbb{Z}$ containing the moduli stack $\mathscr{A}_{g,d}$ of abelian varieties with a polarization of degree d as a dense open substack. The main features of the stack $\overline{\mathscr{A}}_{g,d}$ are that (i) over $\mathbb{Z}[1/d]$ it is log smooth (i.e. has toroidal singularities), and (ii) there is a canonical extension of the kernel of the universal polarization over $\mathscr{A}_{g,d}$ to $\overline{\mathscr{A}}_{g,d}$. The stack $\overline{\mathscr{A}}_{g,d}$ is obtained by a certain "rigidification" procedure from a solution to a moduli problem. In the case $d = 1$ the stack $\overline{\mathscr{A}}_{g,1}$ is equal to the normalization of the main component in Alexeev's compactification. In the higher degree case, our study should be viewed as a higher dimensional version of the theory of generalized elliptic curves introduced by Deligne and Rapoport.