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The Geometry of
some special
Arithmetic Quotients



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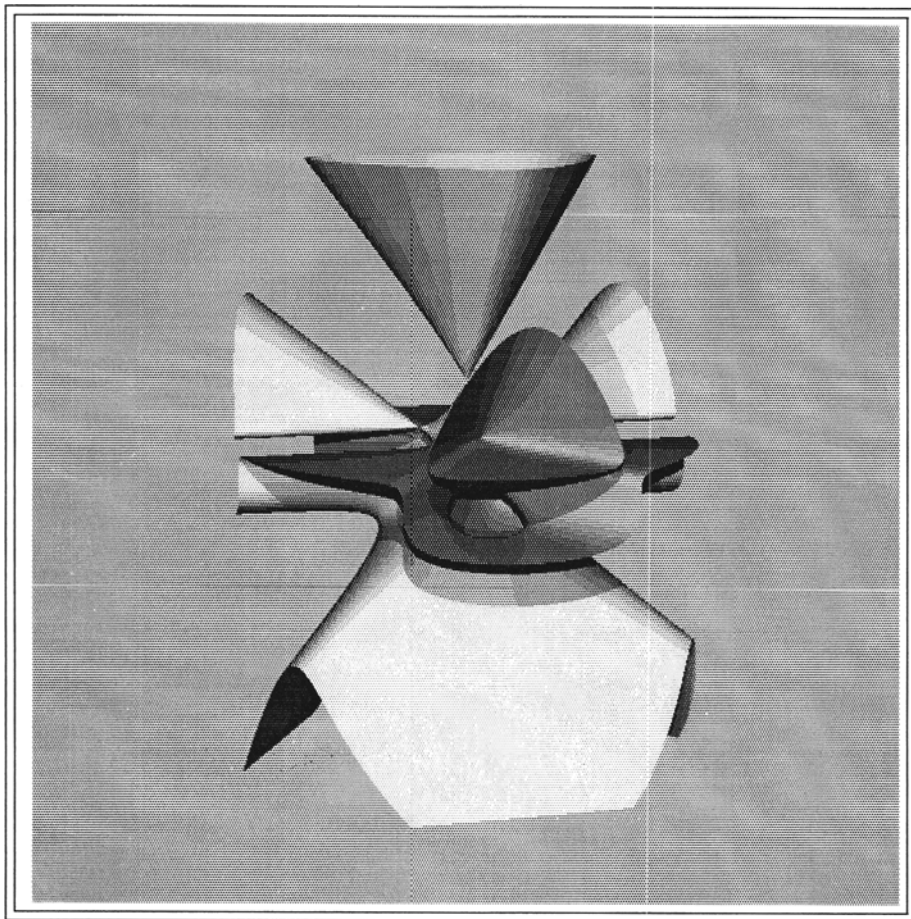
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A space section of the invariant quintic \mathcal{I}_5

More pictures in living color are available at the WWW site:

<http://www.mathematik.uni-kl.de/~wwagag/Galerie.html>

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