Grundlehren der mathematischen Wissenschaften 337

A Series of Comprehensive Studies in Mathematics

Vladimir G. Maz'ya
Tatyana O. Shaposhnikova

Theory of Sobolev Multipliers

With Applications to Differential and Integral Operators

Springer

Vladimir Maz'ya

Department of Mathematical Sciences
M&O Building
University of Liverpool
Liverpool L69 3BX
UK

and

Department of Mathematics
Linköping University
SE-581 83 Linköping
Sweden
vlmaz@mai.liu.se

Tatyana Shaposhnikova

Department of Mathematics
Linköping University
SE-581 83 Linköping
Sweden
tasha@mai.liu.se

ISBN: 978-3-540-69490-8 e-ISBN: 978-3-540-69492-2

Library of Congress Control Number: 2008932182

Mathematics Subject Classification Numbers (2000): 26D10, 46E25, 42B25

Cover design: WMX Design GmbH, Heidelberg

Printed on acid-free paper

springer.com

Contents

Part II Applications of Multipliers to Differential and Integral Operators

Introduction

'I never heard of "Uglification," Alice ventured to say. 'What is it?''

Lewis Carroll,
"Alice in Wonderland"

Subject and motivation. The present book is devoted to a theory of multipliers in spaces of differentiable functions and its applications to analysis, partial differential and integral equations. By a multiplier acting from one function space S_1 into another S_2, we mean a function which defines a bounded linear mapping of S_1 into S_2 by pointwise multiplication. Thus with any pair of spaces S_1, S_2 we associate a third one, the space of multipliers $M(S_1 \to S_2)$ endowed with the norm of the operator of multiplication. In what follows, the role of the spaces S_1 and S_2 is played by Sobolev spaces, Bessel potential spaces, Besov spaces, and the like.

The Fourier multipliers are not dealt with in this book. In order to emphasize the difference between them and the multipliers under consideration, we attach Sobolev's name to the latter. By coining the term Sobolev multipliers we just hint at various spaces of differentiable functions of Sobolev's type, being fully aware that Sobolev never worked on multipliers. After all, Fourier never did either.

Sobolev multipliers arise in many problems of analysis and theories of partial differential and integral equations. Coefficients of differential operators can be naturally considered as multipliers. The same is true for symbols of more general pseudo-differential operators. Multipliers also appear in the theory of differentiable mappings preserving Sobolev spaces. Solutions of boundary value problems can be sought in classes of multipliers. Because of their algebraic properties, multipliers are suitable objects for generalizations of the basic facts of calculus (theorems on implicit functions, traces and extensions, point mappings and their compositions etc.) Moreover, some basic operators

V.G. Maz'ya, T.O. Shaposhnikova, *Theory of Sobolev Multipliers*,
Grundlehren der mathematischen Wissenschaften 337,

of harmonic analysis, like the classical maximal and singular integral operators, act in certain classes of multipliers.

We believe that the calculus of Sobolev multipliers provides an adequate language for future work in the theory of linear and nonlinear differential and pseudodifferential equations under minimal restrictions on the coefficients, domains, and other data.

Before the 1970s, the word *multiplier* was usually associated with the name of Fourier, and a deep theory of L_p-Fourier multipliers created by Marcinkiewicz, Mikhlin, Hörmander *et al* was quite popular. As for the multipliers preserving a space of differentiable functions, only a few isolated results were known (Devinatz and Hirschman [DH], Hirschman [Hi1], [Hi2], Strichartz [Str], Polking [Pol1], Peetre [Pe2]), while the multipliers in pairs of such spaces were not considered at all.

The first (and the only one for the time being) attempt to work out a more or less comprehensive theory of multipliers acting either in one or in a pair of spaces of Sobolev type was undertaken by the authors in the late 1970s and early 1980s [Maz10], [Maz12], [MSh1]–[MSh16]. Results of that theory were collected in our monograph "Theory of Multipliers in Spaces of Differentiable Functions" (Pitman, 1985) [MSh16]. During the last two decades, we continued to work in the area, adding new results and developing further applications [Sh2]–[Sh14], [MSh17]–[MSh23]. We wish to reflect the present state of our theory in this book. An essential part of the aforementioned monograph is also included here.

No results concerning multipliers in spaces of analytic functions are mentioned in what follows, in contrast to [MSh16]. To describe progress in this area achieved during the last twenty five years would require a disproportionate growth of the book.

Structure of the book. The book consists of two parts. Part I is devoted to the theory of multipliers and covers the following topics:

- Trace inequalities
- Analytic characterization of multipliers
- Relations between spaces of Sobolev multipliers and other function spaces
- Maximal subalgebras of multiplier spaces
- Traces and extensions of multipliers
- Essential norm and compactness of multipliers
- Miscellaneous properties of multipliers (spectrum, composition and implicit function theorems, point mappings preserving Sobolev spaces, etc.)

In Part II we dwell upon several applications of this theory. Their list is as follows:

- Continuity and compactness of differential operators in pairs of Sobolev spaces
- Multipliers as solutions to linear and quasilinear elliptic equations

- Higher regularity in the single and double layer potential theory for Lipschitz domains
- Regularity of the boundary in L_p-theory of elliptic boundary value problems
- Singular integral operators in Sobolev spaces

Each chapter starts with a short introductory outline of the included material.

Readership. The volume is addressed to mathematicians working in functional analysis and in the theories of partial differential, integral, and pseudodifferential operators. Prerequisites for reading this book are undergraduate courses in these subjects.

Acknowledgments. V. Maz'ya was partially supported by the National Science Foundation (Grant DMS-0500029, USA) and EPSRC (Grant EP/F005563/1, UK). T. Shaposhnikova gratefully acknowledges support from the Swedish Natural Science Research Council (VR).