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Operator Algebras Generated by Commuting Projections: A Vector Measure Approach

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For Margit, Simon and Sandra; without their constant encouragement and support these notes would never have eventuated.

PREFACE

In the summer semester of 1997 Professor Jim Cooper invited me to give an advanced set of lectures at the Honours/Masters level in the Mathematisches Institut of the Johannes Kepler Universität in Linz, Austria. He left the choice of topic up to me; his only request was that the topic should be of an *interdisciplinary nature* since the students already had a good background in such *individual* courses as algebra, linear algebra, real and complex analysis, functional analysis and measure theory, etc.. The content of this book is essentially an expanded version of the lectures given in Linz. The material was chosen in an attempt to illustrate to advanced students that it is indeed possible to present lecture courses within their mathematical reach which form a connecting bridge between many of the specialized courses that they have already had and such that it "all comes together."

In addition to being able to absorb a body of mathematical knowledge (hopefully developed in a systematic and coherent way), students at this level should also become accustomed to the methodology of mathematical research. They should be able to go to libraries and consult research books and articles, extract from these the relevant information, do some independent thinking, come to the realization that not all problems have instant solutions, etc.. Accordingly, there are many references to the mathematical literature (which the reader is expected to follow up), both in the text and in the various exercises. The exercises are a mixture of fairly routine ones (indicated by [*]) to somewhat more challenging ones and form an integral part of the notes. This book is surely not a pure research book on the topic; for this we refer to the excellent monographs [13] and [15], for example. It is more of a hybrid and, for this reason, definitions and statements of results are carefully formulated and referenced, examples are included to illustrate various points, and many of the proofs are quite detailed as they are designed for the working student and future researcher, and not (necessarily) current experts. At the same time, several of the chapters contain a significant amount of material which may also interest current researchers in the area. Moreover, any reader who achieves a firm grasp of the material is well placed to begin serious research in the general area of operator theory alluded to in these notes, especially in some of the more recent directions. I have here in mind two general areas. Firstly, there is the extension of the theory to the setting of non-normable spaces, where genuine new phenomena arise which are not present in the Banach space setting. Even though much has already been achieved in this direction in the past 20-30 years (see the works of P.G. Dodds, C.K. Fong, J. Junggeburth, S. Okada, B. de Pagter, W.J. Ricker, A. Shuchat, W.V. Smith in the Appendix, and of C.

Ionescu Tulcea, F.Y. Maeda, H.H. Schaefer, B.J. Walsh listed in the bibliography of [15]), there are still several important problems which remain unresolved. The other area is in the direction of harmonic analysis and differential operators in Euclidean L^p-spaces, which often generate families of commuting projections based on algebras or δ-rings of sets rather than σ-algebras of sets; see the works of E. Albrecht, G.B. Folland, G. Gaudry, G.E. Huige, J. Locker, G. Mockenhaupt, M.A. Shobov, W.V. Smith, H.J. Sussmann, I.P. Syroid in the Appendix, and of V.E. Ljance, V.A. Marčenko, M.A. Naimark, B.S. Pavlov, J.T. Schwartz listed in the bibliography of [15]. Such families of projections are typically not uniformly bounded and so will not lead to a Borel functional calculus of the type usually associated with a spectral operator. New techniques will be needed to analyze the operator algebras that such families of projections generate.

Many of the results presented are classical so I have not attempted to record the source of every item. References are not always to the original source, but often to more recent works where further references can be found. The absence of a reference does not necessarily imply originality on my part.

The reader is assumed to have a basic grasp of standard undergraduate courses in algebra, linear algebra, set theory (manipulation), topology, functional analysis, measure theory and integration. Since not all of the readers will have a common base of knowledge in this regard, and for the reason of self-containment, some of these basic notions and facts are included. This is especially true of those having a direct bearing on the subject matter. More specialized material (eg. measure algebras, vector measures and integration, Stone spaces, aspects of operator algebras, functional calculi, etc.) is developed along the way, but only to the extent required for these notes.

In an Appendix at the end of the text I have attempted to form an extensive bibliography of research articles in the general area of spectral operators and Boolean algebras of projections which have appeared *since* 1979. For articles prior to 1979 we refer the interested reader to the excellent bibliographies in [13] and [15]. Some relevant papers prior to 1979 have also been included, provided they do not occur in [13] or [15]. The reason for this Appendix is two-fold. First, it is always useful for any student and/or researcher to have access to such extensive and up-to-date bibliographies. Second, and perhaps more important, I wish to illustrate to students and future researchers that this is an *active* area of modern research. This can be seen not only from the number of articles and their diversity, but also from the number of mathematicians who have contributed to the area.

Special thanks go to my colleagues and friends J.B. Cooper, K. Kiener, E. Matoušková and C. Stegall from Linz. Their encouragement, attention, assistance and above all, their patience, were remarkable. To all of my many colleagues and friends over the past years who have, at various stages, listened to my thoughts and ramblings on this topic (both directly and indirectly) and who have made helpful suggestions (both positive and negative), I especially wish to thank E. Albrecht, R.G. Bartle, I.D. Berg, E. Berkson, P.G. Dodds, I. Doust, D.H. Fremlin, T.A. Gillespie, D. Hadwin, B.R.F. Jefferies, I. Kluvánek, H.P. Lotz, A. McIntosh,

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Sydney; June, 1999

Contents

INTRODUCTION

One of the fundamental facts learnt in linear algebra courses is a basic structural result referred to as the Jordan decomposition theorem. Namely, in a finite dimensional vector space X every linear map $T : X \to X$ can be decomposed as $T = S + N$, where S is a *diagonalizable operator* (i.e. with respect to a suitable basis of X it is similar to a diagonal operator) and N is a *nilpotent operator* (i.e. the spectrum $\sigma(N)$, of N, consists just of $\{0\}$ or, equivalently, $N^k = 0$ for some non-negative integer k) satisfying $SN = NS$. The operator S is called the *scalar part* of T and N is called the *radical part* of T. In particular, S has a representation of the form

$$S = \sum_{j=1}^{r} \lambda_j E_j \,, \tag{1}$$

where $\sigma(S) = \sigma(T) = \{\lambda_j\}_{j=1}^r$ consists of the distinct eigenvalues of S and $\{E_j\}_{j=1}^r$ is a family of non-zero projections (i.e. $E_j^2 = E_j$) with $\sum_{j=1}^r E_j = I$ (the identity operator on X) and satisfying $E_jE_k = 0 = E_kE_j$ whenever $j \neq k$. So, the study of such scalar operators S reduces to a study of the family of much simpler operators $\lambda_j E_j$, for $1 \leq j \leq r$. In fact, if $X_j = E_jX$ is the range of E_j, then the family of vector subspaces $\{X_j\}_{j=1}^r$ has the properties that $X_j \cap X_k = \{0\}$ for $j \neq k$, that $X_1 \oplus \ldots \oplus X_r = X$ and that $SX_j \subseteq X_j$, for $1 \leq j \leq r$. In particular, S restricted to X_j (which is the same as $\lambda_j E_j$ restricted to X_j) acts like $\lambda_j I_j$ in X_j, where $I_j : X_j \longrightarrow X_j$ is the identity operator. For an elegant and succinct account of this topic in terms of linear operators (rather than the usual matrix approach) we refer to [14; Chapter VII, Sections 1 & 2].

What happens if X is infinite dimensional and the linear operator S is continuous? Consider first the case when X is a Hilbert space. If S is *compact* and *normal* (or *selfadjoint*), then the classical spectral theorem of D. Hilbert asserts that $\sigma(S) = \{0\} \cup \{\lambda_j\}_{j=1}^\infty$ is a countable set in $\mathbb{C}$ (or $\mathbb{R}$) with $\lim_{n\to\infty} \lambda_n = 0$ (in the case when $\sigma(S)$ is infinite) and S has a representation of the form (compare with (1))

$$S = \sum_{j=1}^{\infty} \lambda_j E_j \,, \tag{2}$$

where the commuting family of non-zero, selfadjoint projections $\{E_j\}_{j=1}^\infty$ is pairwise disjoint and satisfies $\sum_{j=0}^\infty E_j = I$; here E_0 is the orthogonal projection of X onto $\{x \in X : Sx = 0\}$.

The series (2) and the series $\sum_{j=0}^{\infty} E_j = I$ both converge in the *strong operator topology.* Removing the compactness requirement on S has the effect that $\sigma(S)$ may no longer be discrete. Indeed, $\sigma(S)$ can then be any compact subset of $\mathbb{C}$ (or $\mathbb{R}$ if S is selfadjoint). Moreover, to every Borel set $A \subseteq \mathbb{C}$ (the σ-algebra of all such sets is denoted by $Bo(\mathbb{C})$) there corresponds a selfadjoint projection $E(A)$ such that $E(\emptyset) = 0$, $E(\mathbb{C}) = I$ and the projections in the range $E(Bo(\mathbb{C}))$ of E satisfy

$$(3) \qquad E(A)E(B) = E(A \cap B) = E(B)E(A), \qquad A, B \in Bo(\mathbb{C}),$$

and

$$(4) \qquad \sum_{n=1}^{\infty} E(A_n) = E(\cup_{n=1}^{\infty} A_n),$$

whenever $\{A_n\}_{n=1}^{\infty} \subseteq Bo(\mathbb{C})$ are pairwise disjoint sets. Of course, the series (4) again converges in the strong operator topology. The condition (4) says that $A \mapsto E(A)$ is a *projection-valued measure* on $Bo(\mathbb{C})$. What is the analogue of (2)? Adopting the naive approach that integrals usually replace sums (in the "limit") suggests that

$$(5) \qquad S = \int_{\mathbb{C}} \lambda \, dE(\lambda) = \int_{\sigma(S)} \lambda \, dE(\lambda),$$

where the operator-valued integral (5) needs to be suitably defined. This turns out to indeed be the case and (5) is a formulation of the classical *spectral theorem* for arbitrary normal (or selfadjoint) operators.

The important features from the abstract point of view are that S is synthesized from a certain family of projections $\{E(A) : A \in Bo(\mathbb{C})\}$ via an integral formula of the type

$$S = \int_{\mathbb{C}} f(\lambda) \, dE(\lambda),$$

where $f(\lambda) = \lambda$, for $\lambda \in \mathbb{C}$. Moreover, the multiplicative property (3) of E implies that

$$S^n = \int_{\mathbb{C}} \lambda^n \, dE(\lambda) = \int_{\mathbb{C}} f(\lambda)^n \, dE(\lambda), \qquad n = 0, 1, 2, \dots,$$

and more generally, that

$$g(S) := \int_{\mathbb{C}} g(\lambda) \, dE(\lambda)$$

for any Borel measurable function $g : \mathbb{C} \longrightarrow \mathbb{C}$ which is bounded on $\sigma(S)$. Actually, $\sigma(S)$ turns out to be the *support* of the measure E. So, all reasonable operators which are "functions of S", that is, operators of the form $g(S)$ for suitable g, are built up from the projections $\{E(A) : A \in Bo(\mathbb{C})\}$.

If we wish to stay within the realm of normal operators, then it is necessary to require $\{E(A) : E \in Bo(\mathbb{C})\}$ to be a selfadjoint family. However, the properties (3) and (4) are independent of selfadjointness and so it is undesirable to require this condition from the outset. Moreover, removing this property is no great restriction. Indeed, the well known Mackey-Wermer theorem asserts that if the integral in (5) exists for an arbitrary projection-valued measure E (with respect to the strong operator topology), then there exists a selfadjoint isomorphism $W : X \to X$ such that the family of commuting projections $\{WE(A)W^{-1} : A \in Bo(\mathbb{C})\}$ consists entirely of selfadjoint projections. So, the infinite dimensional Hilbert space analogue of a scalar operator (still called a scalar operator) is any continuous linear operator S which is similar to a normal operator, in which case it has an integral representation of the form (5) for some projection-valued measure E defined on $Bo(\mathbb{C})$. The analogue of a nilpotent operator N is still one which satisfies $\sigma(N) = \{0\}$. However, in infinite dimensional spaces this becomes equivalent to $\lim_{n\to\infty} \|N^n\|^{1/n} = 0$, rather than to some power of N being 0; such operators are called *quasinilpotent.* So, a natural class of continuous linear operators in an infinite dimensional Hilbert space which corresponds to the familiar class of all linear operators in a finite dimensional space, consists of those operators T which have a decomposition

$$(6) \qquad T = S + N = \int_{\sigma(T)} \lambda \, dE(\lambda) + N,$$

where S is a scalar operator and N is a quasinilpotent operator satisfying $SN = NS$. In this formulation we see that even the Hilbert space structure of X is no longer crucial; the definitions of a scalar operator and quasinilpotent operator make perfectly good sense in a general Banach space X. In this setting, operators T of the form (6) are called *spectral operators.* This important class of operators, initiated by N. Dunford in the late 1940's and early 1950's, has undergone intense research ever since.

The aim of these notes is to concentrate on certain particular aspects of the theory of scalar operators, especially in the *Banach space setting*, where the results and methods differ significantly from those in the Hilbert space setting. As discussed above, the central notion is the family of projections $\mathcal{B} = \{E(A) : A \in Bo(\mathbb{C})\}$, the so called *resolution of the identity*, from which the scalar operator S is synthesized. However, to insist on indexing the projections in $\mathcal{B}$ by elements of $Bo(\mathbb{C})$ is, from the theoretical and practical viewpoint, both unnecessary and unduly restrictive. So, the basic concept throughout will be that of a family of commuting projections $\mathcal{B}$, assumed to form a *Boolean algebra* but otherwise not indexed in any particular way. Since we will be interested in those operators which can be "built up" from the elements of the Boolean algebra $\mathcal{B}$, it is natural to require the linear span of $\mathcal{B}$ to be an *algebra* (not just a vector space) and, since some limiting procedures will have to be involved (to pass from sums to integrals, for example), it will also be necessary to take the *closure* of this linear span with respect to some suitable topology. Moreover, to have any hope of identifying elements which arise as some sort of limit from expressions of the form $\sum_{j=1}^{n} \mu_j E_j$, where $\mu_j \in \mathbb{C}$ and $E_j E_k = 0 = E_k E_j$ if $j \neq k$, it is also a necessity to require $\sup\{\|E\| : E \in \mathcal{B}\}$ to be finite; this condition is automatic if $\mathcal{B}$ consists of selfadjoint projections in a Hilbert space, but not in general.

So, we arrive at the following setting: given is a Banach space X and a commutative, unital subalgebra $\mathcal{U}$ (of continuous linear operators on X) which is closed with respect to some topology and is generated by some Boolean algebra of projections $\mathcal{B}$ (assumed to be uniformly bounded). Our main purpose is to investigate, systematically and in detail, the theory of such operator algebras and to attempt to answer various natural questions. As a sample, we will consider the following problems.

(i) Is it possible to give a concrete description of the elements of $\mathcal{U}$ in terms of those from $\mathcal{B}$? The answer will depend on various factors; the properties of the underlying Banach space X, the topology used in $\mathcal{U}$, and on certain properties of $\mathcal{B}$ itself. This question is the central theme of Chapter III, where the uniform operator topology is considered, and of Chapter V, where the strong and weak operator topologies are relevant.

(ii) Are the elements of $\mathcal{U}$ all of the form $g(S)$ for suitable functions g and some scalar operator S? The important ingredients here turn out to be the "size" of $\mathcal{B}$ and certain properties of the Banach space X. One of the main results will be to show that the answer is affirmative if the Boolean algebra $\mathcal{B}$ is complete in a certain sense and if X is separable. This forms the core of Chapter VI and is a far reaching extension of the well known fact that every strongly closed Boolean algebra of selfadjoint projections in a separable Hilbert space is the resolution of the identity of some selfadjoint operator.

(iii) Are there other descriptions of the elements of $\mathcal{U}$ with a more algebraic flavour? For instance, if X is a Hilbert space, then a classical result due to J. von Neumann provides a positive answer in terms of the bicommutant of $\mathcal{B}$ (provided that $\mathcal{B}$ consists of selfadjoint projections). Other descriptions are known in terms of the lattice of closed, $\mathcal{B}$-invariant subspaces of X. A detailed discussion of this topic is presented in Chapter VII.

Questions such as those above, and many more, were considered by N. Dunford and others. Several of the major results (but, certainly not all) concerning such operator algebras can be found in two penetrating papers by W.G. Bade [1,2]. These results, and others, are well documented in [13] and [15], for example. Anyone who spends time reading these monographs will realize immediately the beautiful combination of methods employed from a variety of areas within mathematics. From algebra we see the theory of partial orders, Boolean algebras and the representation results of M.H. Stone (as a sample), from functional analysis there is Banach algebra theory, functional calculi, Banach space geometry, weak and weak-star topologies, Alaoglu's theorem and so on, from measure theory we have the Riesz representation theorem, the Radon-Nikodym theorem, the Hahn decomposition theorem, operator-valued integrals and so on, from topology there occur various disconnected spaces, Urysohn's extension theorem, the Stone-Čech compactification, etc. etc.. So, there is no question that we are dealing with an "interdisciplinary topic".

In discussing commutative operator algebras which are uniform operator closed it is natural to employ Banach algebra techniques (as is the case in [15]). However, such methods are not always suitable to describe the *strongly* or *weakly* closed algebra generated by a Boolean algebra of projections. One of our main goals is to systematically employ the methods of *vector measures and integration theory* (developed in Chapter I to the extent needed for our purposes) to represent this algebra as an L^1-space of a *spectral measure.* Once this representation theory is available many of the results alluded to above are easy and natural consequences. In particular, our approach yields proofs of several of the well known theorems in the area which are quite different to the proofs given in [15].

That vector measure techniques can be employed at all relies on the fact that any Boolean algebra of projections $\mathcal{B}$ (with suitable completeness properties) can be realized as the range of a spectral measure defined on the *Baire or Borel sets* of the Stone space of $\mathcal{B}$. This subtle interplay between Boolean algebras of projections and spectral measures, which plays a crucial and unifying role throughout these notes, is carefully developed in Chapter IV. To fully appreciate this subtle connection it is necessary to first consider general Boolean algebras (i.e. not necessarily consisting of projections on some Banach space) and their representation via the *closed-open* subsets of some totally disconnected, compact Hausdorff space. It turns out that the σ-algebra generated by these closed-open sets is precisely the family of *Baire sets.* Typically, the Baire sets form a *proper* sub-σ-algebra of the σ-algebra of all *Borel sets.* All of these features (and more) form the subject matter of Chapter II.

In conclusion, I wish to make it clear that the material presented here forms a personal choice of topics taken from a rather extensive area of research. I have not even attempted to touch on the theory of *spectral* operators, unbounded operators of scalar type, multiplicity theory, sums and products of commuting spectral operators, and so on. For this I refer the interested reader to [13], [15] and to the vast research literature on these topics which has appeared since the publication of [13] and [15], most of which is recorded in the Appendix.