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Positive Polynomials, Convex Integral Polytopes, and a Random Walk Problem

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POSITIVE POLYNOMIALS, CONVEX INTEGRAL POLYTOPES, AND A RANDOM WALK PROBLEM

PROSPECTUS

This monograph concerns itself with results and interconnections in a number of areas; these include positive polynomials, a class of special random walk problems on the lattice $\mathbf{Z}^d$ (d an integer), convex integral polytopes (that is, convex polytopes in $\mathbf{R}^d$ all of whose vertices are lattice points), reflection groups, and commutative algebra. Techniques include those of functional analysis (especially Choquet theory), ordered rings, commutative algebra, and convex analysis. The central problems arise from special actions of tori on C*-algebras, and many of the results in the other areas yield results back at the C* level; the translation is implemented by means of ordered K_0 (of the fixed point C*-algebras). These connections are (roughly) described in Table 1. Since the techniques and results deal with a number of disparate fields, we shall develop the material in each area in considerable detail, as not everyone will be familiar with all of the topics discussed.

This prospectus is intended to outline the various interconnections; topics which may not be familiar to the reader will be introduced in the body of the text. In this prospectus I am trying to make the case that there is a lot of interesting mathematics occurring here, and that the scope for further research is vast.

The motivating problem arises from the classification and description of invariants arising from "xerox" type actions of tori on C*-algebras. Specifically, let $\pi:\mathbf{T} \longrightarrow U(n,C)$ be an n-dimensional representation of the d-torus $\mathbf{T}$; let $A = \otimes M_nC$ (n fixed) be the infinite tensor product of $n \times n$ matrix algebras, and define $\alpha: \mathbf{T} \longrightarrow \mathrm{Aut}(A)$ via $\alpha(g) = \otimes \mathrm{Ad}\pi(g)$. Then the fixed point algebra of A under the action of α, $A^{\mathbf{T}}$, is of great interest. The classification of such objects is done by means of ordered $K_0(A^{\mathbf{T}})$. Consequences of this include the parameterization of all primitive ideals of the fixed point algebras, including explicit generating sets for these ideals. Moreover, the algebraic structures developed herein yield invariants (essentially complete and computable) for $A^{\mathbf{T}}$.

In order to compute the ordering on $K_0(A^{\mathbf{T}})$ (which is the crucial ingredient of this invariant), one has to solve problems of the following type. Given (Laurent) polynomials P and f in several (real) variables, with the coefficients of P being non-negative, determine necessary and sufficient conditions on f for there to exist an integer n so that $P^n f$ also has no negative coefficients. (Reference [H1] is primarily devoted to this, and the solution is given in [H2].) Here P corresponds to the character of π.

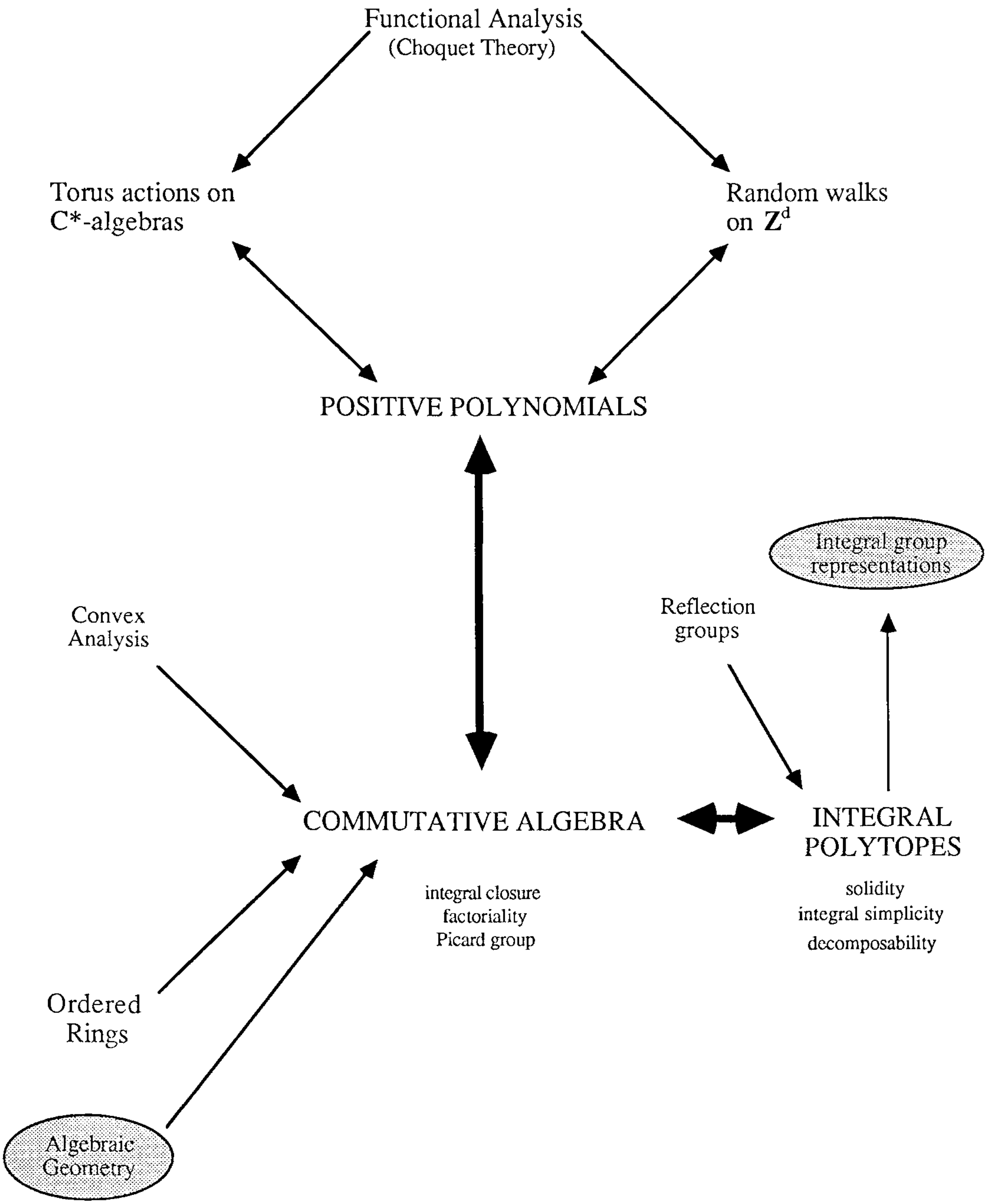
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Table 1

For example, consider what happens when $d = 1$, that is, P and f are polynomials in a single variable x. Provided P is not trivial and has no gaps (this means that if x^a and x^c appear in P, and $a < b < c$ with b an integer, then x^b appears in P), the conditions on f are otherwise independent of the choice of P; f is strictly positive as a function on the interval $(0,\infty)$. (This can be deduced from a result of Meissner [M], and is also proved in [H1]). Similarly, whether $P = x^3 + x + 1$ or $P = x^3 + 2x + 4$ (or any $P = x^3 + dx + e$, with d and e greater than 0), the conditions on f depend only on the set of supporting monomials in P. In this case, the conditions are that f be strictly positive on the interval $(0,\infty)$ and that the coefficient of $x^{\deg f-1}$ be non-negative [H2; Example V.2].

In several variables, the conditions on f (given P) are so complicated [H2], that it is not clear that this principal applies; in other words, if P_o and P_1 are polynomials with only positive coefficients having identical sets of supporting monomials, is it true that $P_o^n f$ has only positive coefficients (for some n) if and only if there exists m so that $P_1^m f$ has as well?

If we translate this (via the inverse Fourier transform) to a question about random walks on $\mathbf{Z}^d$, then this would seem to be counter-intuitive, Let ν_f be a signed measure of finite support (the correspondence between f and ν_f being given by $f = \sum \nu_f(w) x^w$) and let μ_o and μ_1 be finitely supported positive measures on $\mathbf{Z}^d$. Is it true that with $*$ denoting convolution, $(\mu_o * \mu_o * ... * \mu_o) * \nu_f$ being positive for some number of μ_o's implies that $(\mu_1 * \mu_1 * ... * \mu_1) * \nu_f$ is positive as well (for a sufficiently large number of μ_1's)? For example, suppose that the support of μ_o and of μ_1 is the set of vertices of the unit square in $\mathbf{Z}^2$, and most of the mass of μ_o is concentrated at $(0,0)$, while most of the mass of μ_1 is at $(1,1)$. It seems unreasonable that the answer to our question should be affirmative, because each measure is distorted in a different direction. Nonetheless, the result *is* true, and is even true with a slight weakening of the hypotheses. This is the principal result of section II, and forms much of the basis for the rest of the paper. The techniques involve a variant of Choquet theory developed in [GH1]) for dimension groups, as well as those of commutative algebra.

This result permits us to define an $AGL(d,\mathbf{Z})$-invariant for integral polytopes, as follows.

Let K be an integral polytope in $\mathbf{R}^d$. Let P be any polynomial with only positive coefficients, whose set of supporting monomials is $K \cap \mathbf{Z}^d$; that is, write $P = \sum \lambda_w x^w$ (here $w = (w(1),w(2),...,w(d))$ is a point in $\mathbf{Z}^d$ and $x^w = x_1^{w(1)} x_2^{w(2)} ... x_d^{w(d)}$), with all of the λ_w being non-negative real numbers such that λ_w is not zero if and only if w belongs to $K \cap \mathbf{Z}^d$. Form the ring $R_P = \mathbf{R}[x^w/P; w \in K \cap \mathbf{Z}^d]$ (this is a subalgebra of $\mathbf{R}[x_i^{\pm 1}, P^{-1}]$) with positive cone generated additively and multiplicatively by $\{x^w/P \mid w \in K \cap \mathbf{Z}^d\}$. If f is a polynomial with f/P^k in R_P for some integer k, then f/P^k belongs to the positive cone if and only if there exists n so that $P^n f$ has no negative coefficients. Let U denote the set of elements $\{a = f/P^k \in R_P | a(r_1,...,r_d) > \varepsilon$ for some ε, but for all strictly positive real $r_1, ..., r_d\}$. This is precisely that set of elements of R_P that are bounded below by a positive rational multiple of the constant function 1 with respect to the ordering. Form $U^{-1}R_P$ (i.e., invert all of the members of the multiplicatively closed set U). The preceding random walk result yields that $U^{-1}R_P$ depends *only on K* (as an ordered ring) and not upon the choice of P. Set $R_K = U^{-1}R_P$; this is easily seen to be an AGL(d,**Z**)-invariant for K. In fact we treat somewhat more general objects arising from arbitrary finite subsets of $\mathbf{Z}^d$, instead of $K \cap \mathbf{Z}^d$.

The obvious thing to do, is to develop a lexicon between (algebraic) properties of R_K and integro-geometric properties of K. We deal with several properties of R_K, e.g., being integrally closed, Cohen-Macauley, or factorial (and for R_P, being regular). In each case, interesting subsidiary results come out. For example, a standard easy to prove fact is that if $\{a_i\}$ is a finite set of positive integers whose greatest common divisor is 1, then all sufficiently large integers are positive integer combinations of the a_i. The higher dimensional analogue is neither obvious nor (apparently) known. If A is a subsemigroup of $\mathbf{Z}^d$ (under addition) such that $A-A = \mathbf{Z}^d$, and b is a lattice point in the interior of the convex hull of A, then there exists an integer m so that for all $n \geq m$, nb belongs to A. Strange phenomena occur along the boundary of the convex hull of A, and therein lies the obstruction to R_{dK} being integral over R_K in general $(d = \dim K)$.

One rather surprising phenomenon occurring only when the dimension is at least three, is that $2(K \cap \mathbf{Z}^d) \neq 2K \cap \mathbf{Z}^d$, i.e., there exists w in $2K \cap \mathbf{Z}^d$ that is not a sum of two (or frequently, an arbitrary number of) elements of $K \cap \mathbf{Z}^d$, even when $K \cap \mathbf{Z}^d - K \cap \mathbf{Z}^d$ generates $\mathbf{Z}^d$ as an abelian group. An integral polytope is called solid when $n(K \cap \mathbf{Z}^d) = nK \cap \mathbf{Z}^d$ for all n, and a local version of this is precisely the criterion (when $K \cap \mathbf{Z}^d - K \cap \mathbf{Z}^d$ generates $\mathbf{Z}^d$) for R_K (and R_P) to be integrally closed (and Cohen-Macauleyness also occurs). It turns out that solidity is frequently easy to verify, as $n(K \cap \mathbf{Z}^d) = nK \cap \mathbf{Z}^d$ is sufficient, as is $K = eK'$ for some $e \geq d$ and K' an arbitrary integral polytope.

One of the principal tools is the Riesz interpolation property; this is satisfied by R_P (when viewed as an ordered abelian group under addition). This allows us to show that every localization of R_P at a prime ideal is also a localization of a monomial algebra, hence is amenable to results of Hochster [Ho].

A first step in determining conditions for factoriality of R_K (and regularity of R_P), and which is of interest in its own right, is to show that all projectives over R_K (arbitrary K) are free. To do this, we use the Legendre transformation from convex analysis. Let $P = \sum \lambda_w x^w$ be a polynomial in d variables with no negative coefficients. Define the map $\Lambda:(\mathbf{R}^d)^{++} \longrightarrow \mathrm{Int}\, K$ via

$$\Lambda(r) = \frac{\sum \lambda_w r^w w}{\sum \lambda_w r^w} = \left(\ldots, \left. \frac{x_i \frac{\partial P}{\partial x_i}}{P} \right|_{x=r}, \ldots \right)$$

(Here, $r^w = r_1^{w(1)} r_2^{w(2)} \ldots r_d^{w(d)}$.) If K has interior, then a very special case of [Ro; Theorem 26.5] yields that Λ is a homeomorphism. This beautiful result, also known to probabilists [N], [INN] is apparently unknown to differential and algebraic topologists (as an informal survey of more than 20 revealed!). So I have included a proof of this result using only elementary techniques from real and complex analysis, functional analysis, and operators on Hilbert space (Appendix E).

The connections between the Legendre transformation and projectives over R_K run as follows. If G is a partially ordered abelian group, a state of G is a positive real-valued group homomorphism. The states of R_K normalized at 1, form a Choquet (in fact, a Bauer) simplex. Among them are point evaluations arising from r in $(\mathbf{R}^d)^{++}$, given by $a \mapsto a(r)$ for a in R_K. It is known that the set of these is dense in the set of pure (= extremal) states, which we denote $d_eS(R_K,1)$. For any choice of P that will generate K, there is a natural map, $\Lambda_P: d_eS(R_K,1) \longrightarrow K$ which when restricted to $(\mathbf{R}^d)^{++}$ (viewed as the set of point evaluation states) is simply Λ. It is known that $d_eS(R_K,1) \backslash (\mathbf{R}^d)^{++}$ is a union of lower dimensional spaces that are each state spaces of a similar type (arising from polynomials in effectively fewer variables than P), and from Λ being a homeomorphism, it follows inductively that Λ_P is as well. Thus $d_eS(R_K,1)$ is homeomorphic to K, so in particular, is contractible.

Now R_K embeds naturally as a ring of continuous (real-valued) functions on $d_eS(R_K,1)$ (this is the standard method of analyzing dimension groups—by representing them as groups of affine functions on the state space), and moreover, everything in R_K that does not admit a zero (as a function) on the compact space $d_eS(R_K,1)$ has an inverse in R_K. Standard arguments (as in [Sw1]) then yield that all projectives are free, as a consequence of contractibility of $d_eS(R_K,1)$.

Factoriality of R_K reduces to regularity of R_P. This turns out to be equivalent (when $K \cap \mathbf{Z}^d - K \cap \mathbf{Z}^d$ generates $\mathbf{Z}^d$ as an abelian group) to the following very strong geometric property for K:

> For every vertex of K, the convex hull of that vertex together with its nearest neighbours in $K \cap \mathbf{Z}^d$ along the edges (1-faces) of K that contain the vertex, is AGL(d,**Z**)-equivalent to the standard solid d-simplex.

There is a very simple test for an integral polytope to be AGL(d,**Z**)-equivalent to the standard simplex—it is simply that the volume be 1/d!. An integral polytope K satisfying the property above will be called integrally simple; this notion is the integral analogue of simplicity for real polytopes. We study integrally simple polytopes in some detail. Their key property is that they admit "local integral coordinates". This may be exploited, for example, to determine all the meet-irreducible order ideals of R_K; this translates back to the determination of all primitive ideals in the original fixed point C* algebra A^T that we discussed in the beginning.

Such a strong property might seem to render this class too restrictive. However, integrally simple polytopes occur in profusion, as the examples in Appendix D demonstrate. Let W be the Weyl group of some compact connected semisimple Lie group. It acts in a natural way on the dual of the maximal torus, i.e., on $\mathbf{Z}^d$. Let a be any lattice point, and let K_a be the convex hull of the W-orbit of a. Regarded (after translation) as inside the sublattice of $\mathbf{Z}^d$ generated by $\{a^g - a \mid g \in W\}$, K_a is integrally simple (with respect to this sublattice) for almost all choices of a; in particular, K_a is integrally simple if a is not on the boundary of the Weyl chamber to which it belongs. This type of result also holds, with restrictions, for some reflection groups.

This result suggests a geometric invariant for general finite group integral representations—given a group acting on $\mathbf{Z}^d$, look at the convex hulls of the orbits, and ask questions about their AGL(d,**Z**)-invariant properties, either individually, or as above en masse. We do not investigate this here (which explains the shaded oval in table 1).

Automorphisms of R_K are of interest. We show that if K is either integrally simple or indecomposable (in the sense of real polytopes, that is, K = K' + K" implies one of K' or K" is real homothetic to K), then any algebra automorphism of R_K is automatically an order-automorphism (section VIII). While this would seem to indicate that the order structure can be done away with (and there is no reason to believe the result fails for more general K), in fact, we use order-theoretic methods and concepts throughout, and these provide us with indispensable tools and points of view. It follows from this result, that for these K, the automorphisms of R_K are of a very special form. Aside from a finite group of symmetries on K, there is a collection of automorphisms associated bijectively with the faithful pure states of R_K, i.e., with $(\mathbf{R}^d)^{++}$. These (dropping the finite top) appear to be the only ones, and this conjecture leads to what seems to be a difficult problem in algebraic geometry, concerning special automorphisms of function fields. Except in the relatively trivial case of d = 1, I could not solve this, explaining the encircling of algebraic geometry in the table.

In connection with indecomposables (or rather decomposables), there is a rather startling relation between decompositions, factorization of polynomials in several variables, and the Picard group of R_P. Let P (with only positive coefficients) be such that its set of exponents arising from supporting monomials is of the form $K \cap \mathbf{Z}^d$, where K is integrally simple. Then R_P is (homologically) regular, and its Picard group, $\text{Pic}(R_P)$ is generated by prime (order) ideals corresponding to faces of codimension 1 in K.

Essentially, the obstructions to $\text{Pic}(R_P)$ being trivial are given by a failure of P to admit a factorization corresponding to a decomposition of K as a sum of other (possibly homothetic) rational polytopes (which need not be simple themselves). In particular, $\text{Pic}(R_P)$ is as large as possible (with K fixed) when P is irreducible (and there is a simple formula in terms of d and the number of maximal faces), and gets smaller as P starts having factorizations. A careful examination of how factorizations of P lead to new relations among the generators of $\text{Pic}(R_P)$ yields that integrally simple polytopes decompose into the known maximal number of indecomposable pairwise non-homothetic rational polytopes (i.e., the theoretical upper bound is attained). (This may be known even in the more general context of simple polytopes, but I could not find any such result in the literature.) The argument indirectly requires the Legendre transformation, of course!

Frequently it happens that $\text{Pic}(R_P) = 0$, which upon translation back to a corresponding fixed point C* algebra $A^{\mathbf{T}}$ yields the bizarre

$$K_0(K_0(A^{\mathbf{T}})) = \mathbf{Z}$$

for this particular fixed point algebra.

Scattered throughout the monograph are elaborations on the theme of the random walk problem. For example, if f is in $\mathbf{R}[x_i^{\pm 1}]$ and there exists Q in $\mathbf{R}[x_i^{\pm 1}]^+$ having no negative coefficients such that the same holds for $P = Qf$, does it follow that there exists n so that $P^n f$ has no negative coefficients as well? Easy examples provide a negative answer. However, if R_P is integrally closed (and necessary and sufficient geometric conditions are determined in section III for this to occur) the answer is affirmative.

Suppose P and Q are in $\mathbf{R}[x_i^{\pm 1}]^+$ and Q is obtained from P by deleting a codimension 1 face from the K corresponding to P (that is, K = cvx Log P). If $Q^n f$ has no negative coefficients for some n (with f in $\mathbf{R}[x_i^{\pm 1}]^+$), does it follow that the same holds for $P^m f$ for some m? Again there are plenty of counter-examples. However, if R_P is homologically regular (which translates more or less to its corresponding K being integrally simple), the answer is yes!

Now suppose that P is a fixed polynomial (in several variables) with only positive coefficients, and let $\{P_1, P_2, \ldots\}$ be a sequence of such so that (i) the monomials appearing in each P_i (with nonzero coefficient) are exactly the same as those of P, and (ii) the set of *all* nonzero coefficients of all the P_i's is bounded above and below (away from zero). For f in $\mathbf{R}[x_i^{\pm 1}]^+$, if $P^n f$ has no negative coefficients for some n, there exists m so that $P_1 P_2 \ldots P_m f$ has the same property. The obvious convergence argument just fails to prove this, but a trick involving the thematic random walk result allows it to be pushed through (the converse is a trivial consequence of the random walk result).

We return occasionally to the motivating fixed point C*-algebras. As was mentioned earlier on, when the corresponding K is integrally simple (e.g., if the representation π of $\mathbf{T}$ is the restriction to the maximal torus of an irreducible character of a compact connected semisimple Lie group with dominant weight not on the boundary of the Weyl chamber), the primitive ideals, their structure, subquotients, and generators, can be determined merely by translating back the results on the meet-irreducible order ideals of R_K (section VII). For those who are still interested, we also give a C* algebraic proof of a result in [H1] dealing with the quotients of $A^{\mathbf{T}}$ corresponding to faces of K (Appendix B).

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