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Summary. The theory of Dirichlet forms, Markov semigroups and associated processes on finite and infinite dimensional spaces is reviewed in an unified way. Applications are given including stochastic (partial) differential equations, stochastic dynamics of lattice or continuous classical and quantum systems, quantum fields and the geometry of loop spaces.

0 Introduction

The theory of Dirichlet forms is situated in a vast interdisciplinary area which includes analysis, probability theory and geometry.

Historically its roots are in the interplay between ideas of analysis (calculus of variations, boundary value problems, potential theory) and probability theory (Brownian motion, stochastic processes, martingale theory).

First, let us shortly mention the connection between the "phenomenon" of Brownian motion, and the probability and analysis which goes with it. As well known the phenomenon of Brownian motion has been described by a botanist, R. Brown (1827), as well as by a statistician, in connection with astronomical observations, T.N. Thiele (1870), by an economist, L. Bachelier (1900), (cf. [455]), and by physicists, A. Einstein (1905) and M. Smoluchowski (1906), before N. Wiener gave a precise mathematical framework for its description (1921-1923), inventing the prototype of interesting probability measures on infinite dimensional spaces (Wiener measure). See, e.g., [394] for the fascinating history of the discovery of Brownian motion (see also [241], [16] for subsequent developments).

This went parallel to the development of infinite dimensional analysis (calculus of variation, differential calculus in infinite dimensions, functional analysis, Lebesgue, Fréchet, Gâteaux, P. Lévy...) and of potential theory.

Although some intimate connections between the heat equation and Brownian motion were already implicit in the work of Bachelier, Einstein and Smoluchowski, it was only in the 30's (Kolmogorov, Schrödinger) and the 40's that the strong connection between analytic problems of potential theory and fine properties of Brownian motion (and more generally stochastic processes) became clear, by the work of Kakutani. The connection between analysis and probability (involving the use of Wiener measure to solve certain analytic problems) as further developed in the late 40's and the 50's, together with the application of methods of semigroup theory in the study of partial differential equations (Cameron, Doob, Dynkin, Feller, Hille, Hunt, Martin, ...).
The theory of stochastic differential equations has its origins already in work by P. Langevin (1911), N. Bernstein (30's), I. Gikhman and K. Ito (in the 40's), but further great developments were achieved in connection with the above mentioned advances in analysis, on one hand, and martingale theory, on the other hand.
By this the well known relations between Markov semigroups, their generators and Markov processes were developed, see, e.g. [162], [160], [207], [208], [209], [276], [463].
This theory is largely concerned with processes with "relatively nice characteristics" and with "finite dimensional state space" E (in fact locally compact state spaces are usually assumed). From many areas, however, there is a demand of extending the theory in two directions:

1) "more general characteristics", e.g. allowing for singular terms in the generators
2) infinite dimensional (and nonlinear) state spaces.

As far as 1) is concerned let us mention the needs of handling Schrödinger operators and associated processes in the case of non smooth potentials, see [70].
As far as 2) is concerned let us mention the theory of partial differential equations with stochastic terms (e.g. "noises"), see, e.g. [201], [28], [37], [38], [129], [127] the description of processes arising in quantum field theory (work by Friedrichs, Gelfand, Gross, Minlos, Nelson, Segal...) or in statistical mechanics, see, e.g. [16], [15], [344], [242]. Other areas which require infinite dimensional processes are the study of variational problems (e.g. Dirichlet problem in infinite dimensions) [278], the study of certain infinite dimensional stochastic equations of biology, e.g. [474], the representation theory of infinite dimensional groups, e.g. [68], the study of loop groups, e.g. [30], [12], the study of the development of interest rates in mathematical finance, e.g. [416], [337], [502].
The theory of Dirichlet forms is an appropriate tool for these extensions. In fact it is central for it to work with reference measures μ which are neither necessarily "flat" nor smooth and in replacing the Markov semigroups on continuous functions of the "classical theory" by Markov semigroups on

$L^2(\mu)$-spaces (thus making extensive use of "Hilbert space methods" [211]).
The theory of Dirichlet forms was first developed by Feller in the 1-dimensional case, then extended to the locally compact case with symmetric generators by Beurling and Deny (1958-1959), Silverstein (1974), Ancona (1976), Fukushima (1971-1980) and others (see, e.g., [244], [258]).(Extensions to non symmetric generators were given by J. Elliott, S. Carrillo-Menendez (1975), Y. Lejan (1977-1982), a.a., see, e.g. [367]).
The case of infinite dimensional state spaces has been investigated by S. Albeverio and R. Høegh-Krohn (1975-1977), who were stimulated by previous analytic work by L. Gross (1974) and used the framework of rigged Hilbert spaces (along similar lines is also the work of P. Paclet (1978)). These studies were successively considerably extended by Yu. Kondratiev (1982-1987), S. Kusuoka (1984), E. Dynkin (1982), S.Albeverio and M.Röckner (1989-1991), N. Bouleau and F. Hirsch (1986-1991), see [39], [147], [278], [367], [230], [172], [465], [234], [235], [236], [237], [238], [239], [256].
An important tool to unify the finite and infinite dimensional theory was provided by a theory developed in 1991, by S. Albeverio, Z.M. Ma and M. Röckner, by which the analytic property of quasi regularity for Dirichlet forms has been shown in "maximal generality" to be equivalent with nice properties of the corresponding processes.
The main aim of these lectures is to present some of the basic tools to understand the theory of Dirichlet forms, including the forefront of the present research. Some parts of the theory are developed in more details, some are only sketched, but we made an effort to provide suitable references for further study.
The references should also be understood as suggestions in the latter sense, in particular, with a few exceptions, whenever a review paper or book is available we would quote it rather than an original reference. We apologize for this "distortion", which corresponds to an attempt of keeping the reference list into some reasonable bounds - we hope however the references we give will also help the interested reader to reconstruct historical developments.
For the same reason, all references of the form "see [X]" should be understood as "see [X] and references therein".

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Summary. In this introductory course we review some of the basic concepts of Mathematical Finance. We start with an account on the thesis of L. Bachelier, which was defended as "Théorie de la Spéculation" in Paris in 1900. We hope that this historic approach gives a good motivation for a critical appreciation of the modern theory.

In section 2 we then present the basic framework of the modern no-arbitrage theory in the simple setting of finite probability spaces Ω.

The celebrated Black-Scholes model, based on geometric Brownian motion, is presented in section 3. It is compared to Bachelier's model, which is based on (arithmetic) Brownian motion.

The first three sections are kept on a relatively low level of technical sophistication. In section 4 we pass to a higher level of technicality and review the general theory of semi-martingale models of financial markets. We discuss in some detail the "fundamental theorem of asset pricing", which establishes the relation between the no-arbitrage theory on the one hand, and martingale theory on the other.

Finally, in section 5 we briefly discuss some of the applications of the fundamental theorem.

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