

Lecture Notes in Mathematics

1816

Editors:

J.-M. Morel, Cachan

F. Takens, Groningen

B. Teissier, Paris

Springer

Berlin

Heidelberg

New York

Hong Kong

London

Milan

Paris

Tokyo

S. Albeverio W. Schachermayer M. Talagrand

Lectures on Probability Theory and Statistics

Ecole d'Eté de Probabilités
de Saint-Flour XXX - 2000

Editor: Pierre Bernard



Springer

Authors

Sergio Albeverio
Institute for Applied Mathematics,
Probability Theory and Statistics
University of Bonn
Wegelerstr. 6
53115 Bonn, Germany
e-mail: albeverio@uni-bonn.de

Walter Schachermayer
Department of Financial and
Actuarial Mathematics
Vienna University of Technology
Wiedner Hauptstraße 8–10/105
1040 Vienna, Austria
e-mail: wschach@fam.tuwien.ac.at

Michel Talagrand
Equipe d'Analyse
Université Paris VI
4 Place Jussieu
75230 Paris Cedex 05
France
e-mail: mit@ccr.jussieu.fr

Editor

Pierre Bernard
Laboratoire de Mathématiques Appliquées
UMR CNRS 6620, Université Blaise Pascal
Clermont-Ferrand, 63177 Aubière Cedex
France
e-mail: pierre.bernard@math.univ-bpclermont.fr

Cover: Blaise Pascal (1623-1662)

Cataloging-in-Publication Data applied for

Bibliographic information published by Die Deutsche Bibliothek

Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie;
detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>

Mathematics Subject Classification (2000):

60-01, 60-06, 60G05, 60G60, 60J35, 60J45, 60J60, 70-01, 81-06, 81T08, 82-01, 82B44, 82D30, 90-01, 90A09

ISSN 0075-8434 Lecture Notes in Mathematics

ISSN 0721-5363 Ecole d'Été des Probabilités de St. Flour

ISBN 3-540-40335-3 Springer-Verlag Berlin Heidelberg New York

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

Springer-Verlag Berlin Heidelberg New York a member of BertelsmannSpringer
Science + Business Media GmbH

<http://www.springer.de>

© Springer-Verlag Berlin Heidelberg 2003

Printed in Germany

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typesetting: Camera-ready \TeX output by the authors

SPIN: 10931677 41/3142/du - 543210 - Printed on acid-free paper

Table of Contents

Part I Sergio Albeverio: Theory of Dirichlet forms and applications

0	Introduction	4
1	Functional analytic background: semigroups, generators, resolvents	7
2	Closed symmetric coercive forms associated with C_0 -contraction semigroups	18
3	Contraction properties of forms, positivity preserving and submarkovian semigroups.....	33
4	Potential Theory and Markov Processes associated with Dirichlet Forms	43
5	Diffusions and stochastic differential equations associated with classical Dirichlet forms	51
6	Applications	64
	References	75
	Index	103

Part II Walter Schachermayer: Introduction to the Mathematics of Financial Markets

1	Introduction: Bachelier's Thesis from 1900	111
2	Models of Financial Markets on Finite Probability Spaces .	127
3	The Binomial Model, Bachelier's Model and the Black-Scholes Model	140

4	The No-Arbitrage Theory for General Processes	153
5	Some Applications of the Fundamental Theorem of Asset Pricing	173
	References	177

Part III Michel Talagrand: Mean field models for spin glasses: a first course

1	Introduction	185
2	What this is all about: the REM	188
3	The Sherrington-Kirkpatrick model at high temperature ..	201
4	The p -spin interaction model	213
5	External field and the replica-symmetric solution	221
6	Exponential inequalities	240
7	Central limit theorems and the Almeida-Thouless line	253
8	Emergence and separation of the lumps in the p -spin interaction model	269
	Bibliography	284

Table of Contents

0	Introduction	4
1	Functional analytic background: semigroups, generators, resolvents	7
1.1	Semigroups, Generators	7
1.2	The case of a Hilbert space	13
1.3	Examples	15
2	Closed symmetric coercive forms associated with C_0-contraction semigroups	18
2.1	Sesquilinear forms and associated operators	18
2.2	The relation between closed positive symmetric forms and C_0 -contraction semigroups and resolvents	24
3	Contraction properties of forms, positivity preserving and submarkovian semigroups	33
3.1	Positivity preserving semigroups and contraction properties of forms- Beurling-Deny formula.	33
3.2	Beurling-Deny criterium for submarkovian contraction semigroups	35
3.3	Dirichlet forms	36
3.4	Examples of Dirichlet forms	37
3.5	Beurling-Deny structure theorem for Dirichlet forms	41
3.6	A remark on the theory of non symmetric Dirichlet forms	42
4	Potential Theory and Markov Processes associated with Dirichlet Forms	43
4.1	Motivations	43
4.2	Basic notions of potential theory for Dirichlet forms	44
4.3	Quasi-regular Dirichlet forms	46
4.4	Association of “nice processes” with quasi-regular Dirichlet forms	46
4.5	Stochastic analysis related to Dirichlet forms	50
5	Diffusions and stochastic differential equations associated with classical Dirichlet forms	51
5.1	Diffusions associated with classical Dirichlet forms	51
5.2	Stochastic differential equations satisfied by diffusions associated with classical Dirichlet forms	55
5.3	The general problem of stochastic dynamics	57

5.4	Large time asymptotics of processes associated with Dirichlet forms	59
5.5	Relations of large time asymptotics with space quasi-invariance and ergodicity of measures	60
6	Applications	64
6.1	The stochastic quantization equation and the quantum fields .	64
6.2	Diffusions on configuration spaces and classical statistical mechanics	68
6.3	Other applications	70
6.4	Other problems, applications and topics connected with Dirichlet forms	71

Summary. The theory of Dirichlet forms, Markov semigroups and associated processes on finite and infinite dimensional spaces is reviewed in an unified way. Applications are given including stochastic (partial) differential equations, stochastic dynamics of lattice or continuous classical and quantum systems, quantum fields and the geometry of loop spaces.

0 Introduction

The theory of Dirichlet forms is situated in a vast interdisciplinary area which includes analysis, probability theory and geometry.

Historically its roots are in the interplay between ideas of analysis (calculus of variations, boundary value problems, potential theory) and probability theory (Brownian motion, stochastic processes, martingale theory).

First, let us shortly mention the connection between the “phenomenon” of Brownian motion, and the probability and analysis which goes with it. As well known the phenomenon of Brownian motion has been described by a botanist, R. Brown (1827), as well as by a statistician, in connection with astronomical observations, T.N. Thiele (1870), by an economist, L. Bachelier (1900), (cf. [455]), and by physicists, A. Einstein (1905) and M. Smoluchowski (1906), before N. Wiener gave a precise mathematical framework for its description (1921-1923), inventing the prototype of interesting probability measures on infinite dimensional spaces (Wiener measure). See, e.g., [394] for the fascinating history of the discovery of Brownian motion (see also [241], [16] for subsequent developments).

This went parallel to the development of infinite dimensional analysis (calculus of variation, differential calculus in infinite dimensions, functional analysis, Lebesgue, Fréchet, Gâteaux, P. Lévy...) and of potential theory.

Although some intimate connections between the heat equation and Brownian motion were already implicit in the work of Bachelier, Einstein and Smoluchowski, it was only in the 30's (Kolmogorov, Schrödinger) and the 40's that the strong connection between analytic problems of potential theory and fine properties of Brownian motion (and more generally stochastic processes) became clear, by the work of Kakutani. The connection between analysis and probability (involving the use of Wiener measure to solve certain analytic problems) as further developed in the late 40's and the 50's, together with the application of methods of semigroup theory in the study of partial differential equations (Cameron, Doob, Dynkin, Feller, Hille, Hunt, Martin, ...).

The theory of stochastic differential equations has its origins already in work by P. Langevin (1911), N. Bernstein (30's), I. Gikhman and K. Ito (in the 40's), but further great developments were achieved in connection with the above mentioned advances in analysis, on one hand, and martingale theory, on the other hand.

By this the well known relations between Markov semigroups, their generators and Markov processes were developed, see, e.g. [162], [160], [207], [208], [209], [276], [463].

This theory is largely concerned with processes with "relatively nice characteristics" and with "finite dimensional state space" E (in fact locally compact state spaces are usually assumed). From many areas, however, there is a demand of extending the theory in two directions:

- 1) "more general characteristics", e.g. allowing for singular terms in the generators
- 2) infinite dimensional (and nonlinear) state spaces.

As far as 1) is concerned let us mention the needs of handling Schrödinger operators and associated processes in the case of non smooth potentials, see [70].

As far as 2) is concerned let us mention the theory of partial differential equations with stochastic terms (e.g. "noises"), see, e.g. [201], [28], [37], [38], [129], [127] the description of processes arising in quantum field theory (work by Friedrichs, Gelfand, Gross, Minlos, Nelson, Segal...) or in statistical mechanics, see, e.g. [16], [15], [344], [242]. Other areas which require infinite dimensional processes are the study of variational problems (e.g. Dirichlet problem in infinite dimensions) [278], the study of certain infinite dimensional stochastic equations of biology, e.g. [474], the representation theory of infinite dimensional groups, e.g. [68], the study of loop groups, e.g. [30], [12], the study of the development of interest rates in mathematical finance, e.g. [416], [337], [502].

The theory of Dirichlet forms is an appropriate tool for these extensions. In fact it is central for it to work with reference measures μ which are neither necessarily "flat" nor smooth and in replacing the Markov semigroups on continuous functions of the "classical theory" by Markov semigroups on

$L^2(\mu)$ -spaces (thus making extensive use of “Hilbert space methods” [211]). The theory of Dirichlet forms was first developed by Feller in the 1-dimensional case, then extended to the locally compact case with symmetric generators by Beurling and Deny (1958-1959), Silverstein (1974), Ancona (1976), Fukushima (1971-1980) and others (see, e.g., [244], [258]). (Extensions to non symmetric generators were given by J. Elliott, S. Carrillo-Menendez (1975), Y. Lejan (1977-1982), a.a., see, e.g. [367]).

The case of infinite dimensional state spaces has been investigated by S. Albeverio and R. Høegh-Krohn (1975-1977), who were stimulated by previous analytic work by L. Gross (1974) and used the framework of rigged Hilbert spaces (along similar lines is also the work of P. Paquet (1978)). These studies were successively considerably extended by Yu. Kondratiev (1982-1987), S. Kusuoka (1984), E. Dynkin (1982), S. Albeverio and M. Röckner (1989-1991), N. Bouleau and F. Hirsch (1986-1991), see [39], [147], [278], [367], [230], [172], [465], [234], [235], [236], [237], [238], [239], [256].

An important tool to unify the finite and infinite dimensional theory was provided by a theory developed in 1991, by S. Albeverio, Z.M. Ma and M. Röckner, by which the analytic property of quasi regularity for Dirichlet forms has been shown in “maximal generality” to be equivalent with nice properties of the corresponding processes.

The main aim of these lectures is to present some of the basic tools to understand the theory of Dirichlet forms, including the forefront of the present research. Some parts of the theory are developed in more details, some are only sketched, but we made an effort to provide suitable references for further study.

The references should also be understood as suggestions in the latter sense, in particular, with a few exceptions, whenever a review paper or book is available we would quote it rather than an original reference. We apologize for this “distortion”, which corresponds to an attempt of keeping the reference list into some reasonable bounds - we hope however the references we give will also help the interested reader to reconstruct historical developments.

For the same reason, all references of the form “see [X]” should be understood as “see [X] and references therein”.

Table of Contents

1	Introduction: Bachelier's Thesis from 1900	111
2	Models of Financial Markets on Finite Probability Spaces .	127
3	The Binomial Model, Bachelier's Model and the Black-Scholes Model	140
4	The No-Arbitrage Theory for General Processes	153
5	Some Applications of the Fundamental Theorem of Asset Pricing	173
	References	177

Summary. In this introductory course we review some of the basic concepts of Mathematical Finance. We start with an account on the thesis of L. Bachelier, which was defended as “Théorie de la Spéculation” in Paris in 1900. We hope that this historic approach gives a good motivation for a critical appreciation of the modern theory.

In section 2 we then present the basic framework of the modern no-arbitrage theory in the simple setting of finite probability spaces Ω .

The celebrated Black-Scholes model, based on geometric Brownian motion, is presented in section 3. It is compared to Bachelier's model, which is based on (arithmetic) Brownian motion.

The first three sections are kept on a relatively low level of technical sophistication. In section 4 we pass to a higher level of technicality and review the general theory of semi-martingale models of financial markets. We discuss in some detail the “fundamental theorem of asset pricing”, which establishes the relation between the no-arbitrage theory on the one hand, and martingale theory on the other.

Finally, in section 5 we briefly discuss some of the applications of the fundamental theorem.

Table of Contents

1	Introduction	185
2	What this is all about: the REM	188
3	The Sherrington-Kirkpatrick model at high temperature ..	201
4	The p -spin interaction model	213
5	External field and the replica-symmetric solution	221
6	Exponential inequalities	240
7	Central limit theorems and the Almeida-Thouless line	253
8	Emergence and separation of the lumps in the p -spin interaction model	269
	Bibliography	284