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Non-Life Insurance Mathematics

An Introduction with Stochastic Processes

Springer

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Preface

To the outside world, insurance mathematics does not appear as a challenging topic. In fact, everyone has to deal with matters of insurance at various times of one's life. Hence this is quite an interesting perception of a field which constitutes one of the bases of modern society. There is no doubt that modern economies and states would not function without institutions which guarantee reimbursement to the individual, the company or the organization for its losses, which may occur due to natural or man-made catastrophes, fires, floods, accidents, riots, etc. The idea of insurance is part of our civilized world. It is based on the mutual trust of the insurer and the insured.

It was realized early on that this mutual trust must be based on science, not on belief and speculation. In the 20th century the necessary tools for dealing with matters of insurance were developed. These consist of probability theory, statistics and stochastic processes. The Swedish mathematicians Filip Lundberg and Harald Cramér were pioneers in these areas. They realized in the first half of the 20th century that the theory of stochastic processes provides the most appropriate framework for modeling the claims arriving in an insurance business. Nowadays, the Cramér-Lundberg model is one of the backbones of non-life insurance mathematics. It has been modified and extended in very different directions and, morever, has motivated research in various other fields of applied probability theory, such as queuing theory, branching processes, renewal theory, reliability, dam and storage models, extreme value theory, and stochastic networks.

The aim of this book is to bring some of the standard stochastic models of non-life insurance mathematics to the attention of a wide audience which, hopefully, will include actuaries and also other applied scientists. The primary objective of this book is to provide the undergraduate actuarial student with an introduction to non-life insurance mathematics. I used parts of this text in the course on basic non-life insurance for 3rd year mathematics students at the Laboratory of Actuarial Mathematics of the University of Copenhagen. But I am convinced that the content of this book will also be of interest to others who have a background on probability theory and stochastic processes and would like to learn about applied stochastic processes. Insurance mathematics is a part of applied probability theory. Moreover, its mathematical tools are also used in other applied areas (usually under different names).

The idea of writing this book came in the spring of 2002, when I taught basic non-life insurance mathematics at the University of Copenhagen. My handwritten notes were not very much appreciated by the students, and so I decided to come up with some lecture notes for the next course given in spring, 2003. This book is an extended version of those notes and the associated weekly exercises. I have also added quite a few computer graphics to the text. Graphs help one to understand and digest the theory much easier than formulae and proofs. In particular, computer simulations illustrate where the limits of the theory actually are.

When one writes a book, one uses the experience and knowledge of generations of mathematicians without being directly aware of it. Ole Hesselager's 1998 notes and exercises for the basic course on non-life insurance at the Laboratory of Actuarial Mathematics in Copenhagen were a guideline to the content of this book. I also benefitted from the collective experience of writing EKM [29]. The knowledgeable reader will see a few parallels between the two books. However, this book is an *introduction* to non-life insurance, whereas EKM assume that the reader is familiar with the basics of this theory and also explores various other topics of applied probability theory. After having read this book, the reader will be ready for EKM. Another influence has been Sid Resnick's enjoyable book about Happy Harry [65]. I admit that some of the mathematical taste of that book has infected mine; the interested reader will find a wealth of applied stochastic process theory in [65] which goes far beyond the scope of this book.

The choice of topics presented in this book has been dictated, on the one hand, by personal taste and, on the other hand, by some practical considerations. This course is the basis for other courses in the curriculum of the Danish actuarial education and therefore it has to cover a certain variety of topics. This education is in agreement with the Groupe Consultatif requirements, which are valid in most European countries.

As regards personal taste, I very much focused on methods and ideas which, in one way or other, are related to renewal theory and point processes. I am in favor of methods where one can see the underlying probabilistic structure without big machinery or analytical tools. This helps one to strengthen intuition. Analytical tools are like modern cars, whose functioning one cannot understand; one only finds out when they break down. Martingale and Markov process theory do not play an important role in this text. They are acting somewhere in the background and are not especially emphasized, since it is the author's opinion that they are not really needed for an introduction to non-life insurance mathematics. Clearly, one has to pay a price for this approach: lack of elegance in some proofs, but with elegance it is very much like with modern cars.

According to the maxim that non-Bayesians have more fun, Bayesian ideas do not play a major role in this text. Part II on experience rating is therefore rather short, but self-contained. Its inclusion is caused by the practical reasons mentioned above but it also pays respect to the influential contributions of Hans Bühlmann to modern insurance mathematics.

Some readers might miss a chapter on the interplay of insurance and finance, which has been an open subject of discussion for many years. There is no doubt that the modern actuary should be educated in modern financial mathematics, but that requires stochastic calculus and continuous-time martingale theory, which is far beyond the scope of this book. There exists a vast specialized literature on financial mathematics. This theory has dictated most of the research on financial products in insurance. To the best of the author's knowledge, there is no part of insurance mathematics which deals with the pricing and hedging of insurance products by techniques and approaches genuinely different from those of financial mathematics.

It is a pleasure to thank my colleagues and students at the Laboratory of Actuarial Mathematics in Copenhagen for their support. Special thanks go to Jeffrey Collamore, who read much of this text and suggested numerous improvements upon my German way of writing English. I am indebted to Catriona Byrne from Springer-Verlag for professional editorial help.

If this book helps to change the perception that non-life insurance mathematics has nothing to offer but boring calculations, its author has achieved his objective.

Thomas Mikosch Copenhagen, September 2003

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Thomas Mikosch Copenhagen, February 2006

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Part II Experience Rating

Guidelines to the Reader

This book grew out of an introductory course on non-life insurance, which I taught several times at the Laboratory of Actuarial Mathematics of the University of Copenhagen. This course was given at the third year of the actuarial studies which, together with an introductory course on life insurance, courses on law and accounting, and bachelor projects on life and non-life insurance, leads to the Bachelor's degree in Actuarial Mathematics. This programme has been successfully composed and applied in the 1990s by Ragnar Norberg and his colleagues. In particular, I have benefitted from the notes and exercises of Ole Hesselager which, in a sense, formed the first step to the construction of this book.

When giving a course for the first time, one is usually faced with the situation that one looks for appropriate teaching material: one browses through the available literature (which is vast in the case of non-life insurance), and soon one realizes that the available texts do not exactly suit one's needs for the course.

What are the prerequisites for this book?

Since the students of the Laboratory of Actuarial Mathematics in Copenhagen have quite a good background in measure theory, probability theory and stochastic processes, it is natural to build a course on non-life insurance based on knowledge of these theories. In particular, the theory of stochastic processes and applied probability theory (which insurance mathematics is a part of) have made significant progress over the last 50 years, and therefore it seems appropriate to use these tools even in an introductory course.

On the other hand, the level of this course is not too advanced. For example, martingale and Markov process theory are avoided as much as possible and so are many analytical tools such as Laplace-Stieltjes transforms; these notions only appear in the exercises or footnotes. Instead I focused on a more intuitive probabilistic understanding of the risk and total claim amount processes and their underlying random walk structure. A random walk is one of the simplest stochastic processes and allows in many cases for explicit calculations of distributions and their characteristics. If one goes this way, one essentially walks along the path of renewal and point process theory. However, renewal theory will not be stressed too much, and only some of the essential tools such as the key renewal theorem will be explained at an informal level. Point process theory will be used indirectly at many places, in particular, in the section on the Poisson process, but also in this case the discussion will not go too far; the notion of a random measure will be mentioned but not really needed for the understanding of the succeeding sections and chapters.

Summarizing the above, the reader of this book should have a good background in probability and measure theory and in stochastic processes. Measure theoretic arguments can sometimes be replaced by intuitive arguments, but measure theory will make it easier to get through the chapters of this book.

For whom is this book written?

The book is primarily written for the undergraduate student who wants to learn about some fundamental results in non-life insurance mathematics by using the theory of stochastic processes. One of the differences from other texts of this kind is that I have tried to express most of the theory in the language of stochastic processes. As a matter of fact, Filip Lundberg and Harald Cramér — two pioneers in actuarial mathematics — have worked in exactly this spirit: the insurance business in its parts is described as a continuous-time stochastic process. This gives a more complex view of insurance mathematics and allows one to apply recent results from the theory of stochastic processes.

A widespread opinion about insurance mathematics (at least among mathematicians) is that it is a rather dry and boring topic since one only calculates moments and does not really have any interesting structures. One of the aims of this book is to show that one should not take this opinion at face value and that it is enjoyable to work with the structures of non-life insurance mathematics. Therefore the present text can be interesting also for those who do not necessarily wish to spend the rest of their lives in an insurance company. The reader of this book could be a student in any field of applied mathematics or statistics, a physicist or an engineer who wants to learn about applied stochastic models such as the Poisson, compound Poisson and renewal processes. These processes lie at the heart of this book and are fundamental in many other areas of applied probability theory, such as renewal theory, queuing, stochastic networks, and point process theory. The chapters of this book touch on more general topics than insurance mathematics. The interested reader will find discussions about more advanced topics, with a list of relevant references, showing that insurance mathematics is not a closed world but open to other fields of applied probability theory, stochastic processes and statistics.

How should you read this book?

Part I deals with collective risk models, i.e., models which describe the evolution of an insurance portfolio as a mechanism, where claims and premiums have to be balanced in order to avoid ruin. Part II studies the individual policies and gives advice about how much premium should be charged depending on the policy experience represented by the claim data. There is little theoretical overlap of these two parts; the models and the mathematical tools are completely different.

The core material (and the more interesting one from the author's point of view, since it uses genuine stochastic process theory) is contained in Part I. It is built up in an hierarchical way. You cannot start with Chapter 4 on ruin theory without having understood Chapter 2 on claim number processes.

Chapter 1 introduces the basic model of collective risk theory, combining claim sizes and claim arrival times. The claim number process, i.e., the counting process of the claim arrival times, is one of the main objects of interest in this book. It is dealt with in Chapter 2, where three major claim number processes are introduced: the Poisson process (Section 2.1), the renewal process (Section 2.2) and the mixed Poisson process (Section 2.3). Most of the material of these sections is relevant for the understanding of the remaining sections. However, some of the sections contain informal discussions (for example, about the generalized Poisson process or renewal theory), which can be skipped on first reading; only a few facts of those sections will be used later. The discussions at an informal level are meant as appetizers to make the reader curious and to invite him/her to learn about more advanced probabilistic structures.

Chapter 3 studies the total claim amount process, i.e., the process of the aggregated claim sizes in the portfolio as a function of time. The order of magnitude of this object is of main interest, since it tells one how much premium should be charged in order to avoid ruin. Section 3.1 gives some quantitative measures for the order of magnitude of the total claim amount. Realistic claim size distributions are discussed in Section 3.2. In particular, we stress the notion of heavy-tailed distribution, which lies at the heart of (re)insurance and addresses how large claims or the largest claim can be modeled in an appropriate way. Over the last 30 years we have experienced major man-made and natural catastrophes; see Table 3.2.18, where the largest insurance losses are reported. They challenge the insurance industry, but they also call for improved mathematical modeling. In Section 3.2 we further discuss some exploratory statistical tools and illustrate them with real-life and simulated insurance data. Much of the material of this section is informal and the interested reader is again referred to more advanced literature which might give answers to the questions which arose in the process of reading. In Section 3.3 we touch upon the problem of how one can calculate or approximate the distribution of the total claim amount. Since this is a difficult and complex matter we cannot come up with complete solutions. We rather focus on one of the numerical methods for calculating this distribution, and then we give informal discussions of methods which are based on approximations or simulations. These are quite specific topics and therefore their space is limited in this book. The final Section 3.4 on reinsurance treaties introduces basic notions of the reinsurance language and discusses their relation to the previously developed theory.

Chapter 4 deals with one of the highlights of non-life insurance mathematics: the probability of ruin of a portfolio. Since the early work by Lundberg [55] and Cramér [23], this part has been considered a jewel of the theory. It is rather demanding from a mathematical point of view. On the other hand, the reader learns how various useful concepts of applied probability theory (such as renewal theory, Laplace-Stieltjes transforms, integral equations) enter to solve this complicated problem. Section 4.1 gives a gentle introduction to the topic "ruin". The famous results of Lundberg and Cramér on the order of magnitude of the ruin probability are formulated and proved in Section 4.2. The Cramér result, in particular, is perhaps the most challenging mathematical result of this book. We prove it in detail; only at a few spots do we need to borrow some more advanced tools from renewal theory. Cramér's theorem deals with ruin for the small claim case. We also prove the corresponding result for the large claim case, where one very large claim can cause ruin spontaneously.

As mentioned above, Part II deals with models for the individual policies. Chapters 5 and 6 give a brief introduction to experience rating: how much premium should be charged for a policy based on the claim history? In these two chapters we introduce three major models (heterogeneity, Bühlmann, Bühlmann-Straub) in order to describe the dependence of the claim structure inside a policy and across the policies. Based on these models, we discuss classical methods in order to determine a premium for a policy by taking into account the claim history and the overall portfolio experience (credibility theory). Experience rating and credibility theory are classical and influential parts of non-life insurance mathematics. They do not require genuine techniques from stochastic process theory, but they are nevertheless quite demanding: the proofs are quite technical.

It is recommended that the reader who wishes to be successful should solve the exercises, which are collected at the end of each section; they are an integral part of this course. Moreover, some of the proofs in the sections are only sketched and the reader is recommended to complete them. The exercises also give some guidance to the solution of these problems.

At the end of this book you will know about the fundamental models of non-life insurance mathematics and about applied stochastic processes. Then you may want to know more about stochastic processes in general and insurance models in particular. At the end of the sections and sometimes at suitable spots in the text you will find references to more advanced literature. They can be useful for the continuation of your studies.

You are now ready to start. Good luck!