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Calderón-Zygmund Operators, Pseudo-Differential Operators and the Cauchy Integral of Calderón

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INTRODUCTION

In a survey article entitled "Recent progress in classical Fourier analysis", Charles Fefferman wrote in 1974 the following: "Commutator Integrals. Let $D \subsetneqq C^1$ be a domain bounded by a C^1 curve Γ. Just as in the case of the unit disc, there is a "Hilbert transform" T defined on functions on Γ which sends the real part $u_{|\Gamma}$ of an analytic function $F = u + iv$ to its imaginary part $v_{|\Gamma}$, and it is natural to ask whether T is bounded on $L^2(\Gamma)$ with respect to the arclength measure on Γ. This question is closely connected to the problem of understanding harmonic measure on Γ, i.e., the probability distribution of the place where a particle undergoing Brownian motion starting at a fixed point $P_0 \in D$ first hits Γ.

In effect, T is an integral operator on functions on R^1, given by the formula

$$Tf(x) = \int_{-\infty}^{\infty} \frac{f(y)dy}{(x-y) + i(A(x) - A(y))}$$

with $A \in C^1(R^1)$. Expanding the denominator of the integrand in a geometric series, we obtain T as an infinite sum of operators

$$T_k f(x) = \int_{-\infty}^{\infty} \frac{(A(x) - A(y))^k}{(x-y)^{k+1}} f(y)dy .$$

T_k is called the kth commutator integral corresponding to $A(x)$.

Commutator integrals also arise naturally when one tries to construct a calculus of singular integral operators to handle differential equations with nonsmooth coefficients. T_0 is just the Hilbert transform, but already the following two results are deep.

Theorem: Let A be a C^1 function on the line then

(A) (Calderón [14], 1965) T_1 is bounded on L^2.

(B) (Coifman and Y. Meyer 1974, still unpublished) T_2 is bounded on L^2.

To prove (A), Calderón used special contour integration arguments which unfortunately do not apply to higher T_k's. Coifman and Meyer modified and built on Calderón ideas to produce a far more flexible proof, which can probably be pushed further in the near future to cover all the T_k's and possibly T itself."

One can now judge the accuracy of Fefferman's prophecy: see for instance introduction to Chapter 6. Several results connected to these commutators have been obtained recently by A.P. Calderón, R.R. Coifman, G. David, MacIntosh and Y. Meyer. They motivated greatly the topics chosen in a course I taught in Washington University during the academic year 1981-1982.

Indeed, these commutators are the most interesting examples, besides classical convolution operators, of a class of operators introduced by R. Coifman and Y. Meyer in [CM2], under the name "Calderón-Zygmund operators" (CZO's). Note that their interest lies both in their applications (see [CCFJR], [C3], and [J]) and in the methods one uses to study them, (see [C1], [CM1], and [CMcIM].

Chapters 0 to 3 consist of background material. There, all the tools used in the following chapters are introduced; among which are the Hardy-Littlewood maximal operator, A_p weights of Muckenhoupt, the Calderón-Zygmund decomposition of a function, the spaces H^∞ and BMO, and the Feffermann-Stein #-function.

Chapter 4 is devoted to general properties of CZO's. The essential one is that their L^2-boundedness implies their L^p boundedness for $p \in]1,\infty[$. Substitute results hold for $p=1$ or $p=+\infty$; namely CZO's are bounded from $H^{1,\infty}$ to L^1 and from L^∞ to BMO.

From results of Chapter 4 one can see that the essential problem when dealing with a candidate for a CZO lies in showing the L^2-boundedness. Let us quote the same paper of Charles Feffermann's.

"When neither Plancherel's theorem nor Cotlar's lemma applies, L^2-boundedness of singular operators presents very hard problems, each of which must (so far) be dealt with on its own terms."

That is still true. Chapter 7 illustrates the difficulty of such problems, while in Chapter 8 are explained several specific techniques which already proved to be useful to solve them.

Chapters 5 and 6 treat some connections between the theory of CZO's and the theories of pseudo-differential operators and of Littlewood-Paley.

Since this work is essentially self-contained and not very long, one can easily guess that many applications have not been included. On the other hand, we hope that it is accessible to a wide variety of mathematicians without any specific advanced knowledge in harmonic analysis. Anyhow one should be familiar with the first three chapters of [S] and [SW], for instance, in order to read this set of notes with more ease.

Finally, it is a pleasure to express my thanks to Yves Meyer who introduced me to Harmonic Analysis while teaching a course in Orsay in 78-79, the notes of which I freely used to prepare mine; to Guido Weiss who constantly encouraged and helped me to give a tentatively definite form to this set of notes, and in fact corrected not to say: rewrote-everything (except the present paragraph) and who was, during this whole academic year, the perfect example of the wonderful hospitality of the Washington University Mathematics Department. Thanks also to Richard Rochberg for his irreverent encouragement and occasional help and to Stephen Semmes, who transformed my original unreadable franglish manuscript into a typist's dream. Finally thanks also to Micki Wilderspin who did, as you see, a marvellous typing job.

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