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R. Conte F. Magri M. Musette
J. Satsuma P. Winternitz

Direct and Inverse Methods in Nonlinear Evolution Equations

Lectures Given at the C.I.M.E. Summer School
Held in Cetraro, Italy, September 5-12, 1999

Editor: Antonio M. Greco



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Preface

This book contains the lectures given at the *Centro Internazionale Matematico Estivo* (CIME), during the session **Direct and Inverse Method in Non Linear Evolution Equations**, held at Cetraro in September 1999.

The lecturers were **R. Conte** of the Service de physique de l'état condensé, CEA Saclay, **F. Magri** of the University of Milan, **M. Musette** of Dienst Theoretical Naturalness, Verite Universities Brussels, **J. Satsuma** of the Graduate School of Mathematical Sciences, University of Tokyo and **P. Winternitz** of the Centre de recherches mathématiques, Université de Montréal.

The courses face from different point of view the theory of the exact solutions and of the complete integrability of non linear evolution equations.

The Magri's lectures develop the geometrical approach and cover a large amount of topics concerning both the finite and infinite dimensional manifolds, Conte and Musette explain as Painlevé analysis and its various extensions can be extensively applied to a wide range of non linear equations. In particular Conte deals with the ODEs case, while Musette deals with the PDEs case. The Lie's method is the main subject of Winternitz's course where is shown as any kind of possible symmetry can be used for reducing the considered problem, and eventually for constructing exact solutions.

Finally Satsuma explains the bilinear method, introduced by Hirota, and, after considering in depth the algebraic structure of the completely integrable systems, presents modification of the method which permits to treat, among others, the ultra-discrete systems.

All lectures are enriched by several examples and applications to concrete problems arising from different contexts. In this way, from one hand the effectiveness of the used methods is pointed out, from the other hand the interested reader can experience directly the different geometrical, algebraical and analytical machineries involved.

I wish to express my appreciation to the authors for these notes, updated to the summer 2002, and to thank all the participants of this CIME session.

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