Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1143

Aloys Krieg

Modular Forms on Half-Spaces of Quaternions

Springer-Verlag
Berlin Heidelberg New York Tokyo

Author

Aloys Krieg
Mathematisches Institut der Westfälischen Wilhelms-Universität
Einsteinstr. 62, 4400 Münster, Federal Republic of Germany

Mathematics Subject Classification (1980): 10D20

ISBN 3-540-15679-8 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-15679-8 Springer-Verlag New York Heidelberg Berlin Tokyo

Library of Congress Cataloging-in-Publication Data. Krieg, Aloys, 1955 – Modular forms on half-spaces of quaternions. (Lecture notes in mathematics; 1143) Bibliography: p. Includes index. 1. Forms, Modular. 2. Quaternions. I. Title II. Series: Lecture notes in mathematics (Springer-Verlag; 1143. QA.L28 no. 1143 520 s 85-17284 [QA243] [512'.522]
ISBN 0-387-15679-8 (U.S.)

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

Table of contents

Introduction

The mathematical background

The theory of elliptic modular forms and functions developed from the investigation of elliptic functions in the 19th century. One proceeds from the upper half

(1) $\mathcal{H} := \{z = x + iy \in \mathbb{C} \; ; \; y > 0\}$

in $\mathbb{C}$. By means of the group

(2) $SL(2;\mathbb{R}) := \{M \in Mat(2;\mathbb{R}) \; ; \; \det M = 1\}$

one can describe the biholomorphic maps of $\mathcal{H}$ onto itself. All these maps are of course given by the transformations

(3) $z \longmapsto M\langle z\rangle := \frac{az+b}{cz+d}$, whenever $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{R})$.

The so-called modular group

(4) $\Gamma := SL(2;\mathbb{Z})$

forms a discrete subgroup of $SL(2;\mathbb{R})$ and plays an essential part.

By a fundamental domain one means a closed set of representatives of $\mathcal{H}$ with respect to the equivalence relation induced by Γ , where every two different interior points are not equivalent. Such a domain can easily be described by

Theorem A. $\mathcal{F} := \{z = x + iy \in \mathcal{H} \; ; \; |x| \leq \frac{1}{2} \; , \; |z| \geq 1\}$ *is a fundamental domain of* $\mathcal{H}$ *with respect to the action of* Γ .

Given $k \in \mathbb{Z}$ a holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is called an *elliptic modular form of weight* k if the following two properties are satisfied:

(5) $$\begin{cases} f(M\langle z\rangle)\,(cz+d)^{-k} = f(z) \quad \text{for all } z \in \mathcal{H} \text{ and } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \, . \\ f \text{ is bounded in every domain } y \geq \rho > 0 \, . \end{cases}$$

By virtue of the boundedness, a FOURIER-expansion at ∞ in the form

(6) $$f(z) = \sum_{t=0}^{\infty} \alpha(t)\, e^{2\pi i t z} \; , \; z \in \mathcal{H} \; ,$$

is obtained. The set $[\Gamma,k]$ of all elliptic modular forms of weight k becomes a $\mathbb{C}$-vector space.

Examples are given by the EISENSTEIN-series

$$(7)\qquad E_k(z) = \sum_{\substack{(c,d)\in\mathbb{Z}\times\mathbb{Z}\\(c,d)\neq(0,0)}} (cz+d)^{-k}\ ,\quad z\in H\ ,$$

which are non-identically vanishing modular forms of weight k, whenever $k > 2$ is even.

The theta-series form another class of examples.

$$(8)\qquad \Theta(z,S) := \sum_{\mathfrak{g}\in\mathbb{Z}^m} e^{\pi i\mathfrak{g}'S\mathfrak{g}z}\ ,\quad z\in H\ ,$$

becomes a modular form of weight $\frac{1}{2}m$, whenever $S = S^{(m)}$ is even, unimodular and positive definite.

By means of the EISENSTEIN-series, the vector space of modular forms can be described in the following well-known way:

<u>Theorem B.</u> a) <u>One has</u> $[\Gamma,k] = \{0\}$, <u>whenever</u> k <u>is negative or odd</u>. <u>If</u> $k \geq 0$ <u>is even then</u>

$$\dim\,[\Gamma,k] = \begin{cases} \left[\frac{k}{12}\right] & ,\ \underline{\text{if}}\ \ k\equiv 2 \bmod 12\ ,\\[2ex] \left[\frac{k}{12}\right]+1 & ,\ \underline{\text{else}}\ ,\end{cases}$$

<u>holds</u>.

b) <u>Let</u> $k \geq 0$ <u>be even, then the products of</u> EISENSTEIN-<u>series</u> $E_4^l E_6^m$, <u>where</u> $l,m\in\mathbb{N}_0$ <u>and</u> $4l+6m = k$, <u>form a basis of</u> $[\Gamma,k]$.

A meromorphic function f on H, which does not possess an essential singularity at ∞, is called an <u>elliptic modular function</u> if

$$(9)\qquad f(M\langle z\rangle) = f(z)\quad\text{for all}\quad z\in H\quad\text{and}\quad M\in\Gamma\ .$$

Let M denote the field of elliptic modular functions. Then

$$(10)\qquad j := \frac{1728\,(60E_2)^3}{(60E_2)^3-(140E_3)^2}$$

belongs to M, and one can derive the fundamental

Theorem C. $\mathcal{M} = \mathbb{C}(j)$.

Especially the transcendence degree of $\mathcal{M}$ over $\mathbb{C}$ equals 1 and every modular function can be represented as a quotient of two modular forms of the same weight.

The theory of elliptic modular forms has been generalized in several respects. Replacing $\mathcal{H}$ by a product of upper half-planes one attains to the theory of HILBERT's or HILBERT-BLUMENTHAL's modular functions.

In the 30's SIEGEL [53,54] investigated the so-called modular functions of degree n , which nowadays are known as SIEGEL's modular functions. In the case $n = 1$ they coincide with the elliptic modular functions.

With regard to this generalization one has to replace $\mathcal{H}$ by SIEGEL's half-space

(1') $$H(n;\mathbb{R}) := \{Z = X + iY \in \mathrm{Mat}(n;\mathbb{C})\ ;\ X=X'\ , Y=Y' \text{ positive definite}\}.$$

Considering the symplectic group

(2') $$Sp(n;\mathbb{R}) := \{M \in \mathrm{Mat}(2n;\mathbb{R})\ ;\ M'JM = J\}\ ,\ J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\ ,$$

instead of $SL(2;\mathbb{R})$, all biholomorphic maps of SIEGEL's half-space onto itself are given by the symplectic transformations

(3') $$Z \longmapsto M\langle Z\rangle := (AZ + B)(CZ + D)^{-1}\ ,\ \text{where}\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n;\mathbb{R})\ ,$$

in view of [54]. The so-called modular group

(4') $$\Gamma_n := Sp(n;\mathbb{Z})$$

forms a discrete subgroup of $Sp(n;\mathbb{R})$, which acts discontinuously on SIEGEL's half-space. By means of MINKOWSKI's reduction theory SIEGEL [54] constructed a fundamental domain:

Theorem A'.

$$\mathcal{F}_n := \left\{ Z = X + iY \in H(n;\mathbb{R})\ ;\ \begin{array}{l} X \bmod 1\ ,\ Y\ \underline{\text{reduced}}\ , \\ |\det(CZ+D)| \geq 1\ \underline{\text{for}}\ M \in \Gamma_n \end{array} \right\}$$

is a fundamental domain of $H(n;\mathbb{R})$ with respect to the action of the modular group Γ_n .

Let $k \in \mathbb{Z}$, $n \geq 2$ and $f : H(n;\mathbb{R}) \longrightarrow \mathbb{C}$ be holomorphic. f is called SIEGEL's modular form of weight k if

$$(5') \qquad f(M\langle Z\rangle)\det(CZ+D)^{-k} = f(Z)$$

holds for all $Z \in H(n;\mathbb{R})$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. KOECHER [35] discovered that any requirement of boundedness is not necessary, whenever $n \geq 2$. A modular form f possesses a FOURIER-expansion of the form

$$(6') \qquad f(Z) := \sum_{T} \alpha(T)\, e^{2\pi i \operatorname{trace} TZ} , \quad Z \in H(n;\mathbb{R}) ,$$

where T runs through the set of all positive semi-definite, semi-integral $n \times n$ matrices.

SIEGEL [54] showed that the vector space $[\Gamma_n,k]$ of all modular forms of weight k has finite dimension and consists only of 0 , if k is negative. But even nowadays it seems to be impossible to compute the exact dimension or, moreover, to state a basis for arbitrarily given k and n . The case $n = 2$ however was solved on the analogy of Theorem B by IGUSA [26] and later by FREITAG [17].

MAASS [43;44] generalized PETERSSON's scalar product to SIEGEL'S modular forms. By virtue of this theory of metrization he succeeded in proving a theorem of representation, which says that $[\Gamma_n,k]$ is generated by POINCARÉ -series, whenever $k > 2n$ is even. Another version of the theorem of representation by means of generalizations of EISENSTEIN-series (7) is due to KLINGEN [32].

Theta-series on SIEGEL's half-space are built in the form

$$(8') \qquad \Theta(Z,S) := \sum_{G \in Mat(m,n;\mathbb{Z})} e^{\pi i \operatorname{trace}(G'SGZ)} , \quad Z \in H(n;\mathbb{R}) .$$

If $S = S^{(m)}$ is even, unimodular and positive definite $\Theta(\cdot,S)$ belongs to $[\Gamma_n,\frac{1}{2}m]$. RESNIKOFF [48,49] and FREITAG [19,20] showed that the theta-series form a basis of $[\Gamma_n,k]$, whenever $0 < k < \frac{1}{2}n$. The meaning of theta-series is emphasized by BÖCHERER's result [5] that the theta-series span the vector space $[\Gamma_n,k]$, whenever $k > 2n$ and $k \equiv 0 \bmod 4$.

Summarizing one has

<u>Theorem B'.</u> a) <u>Every</u> SIEGEL'<u>s</u> <u>modular form of weight</u> $k < 0$ <u>vanishes identically and</u> $\dim[\Gamma_n,k] < \infty$ <u>holds for</u> $k \geq 0$.

b) <u>If</u> $0 < k < \frac{1}{2}n$ <u>the theta-series form a basis and for</u> $k > 2n$, $k \equiv 0 \bmod 4$ <u>they span</u> $[\Gamma_n,k]$.

c) POINCARÉ - <u>and</u> EISENSTEIN-<u>series span</u> $[\Gamma_n,k]$, <u>whenever</u> $k > 2n$ <u>is even</u>.

If $n \geq 2$ a meromorphic function f on $H(n;\mathbb{R})$ satisfying

(9') $$f(M\langle Z\rangle) = f(Z) \quad \text{for all} \quad Z \in H(n;\mathbb{R}) \, , \; M \in \Gamma_n$$

is called SIEGEL's modular function. The set $\mathcal{M}_n$ of all modular functions on $H(n;\mathbb{R})$ forms a field of transcendence degree $\frac{1}{2}n(n+1)$. First it was shown by means of deep-rooted results of the algebraic geometry and later on by SIEGEL [57] using classic methods that every modular function can be represented as the quotient of two modular forms of the same weight. Finally one has the fundamental

Theorem C'. There are SIEGEL's modular functions $f_o,\dots,f_h$, $h = \frac{1}{2}n(n+1)$, satisfying

$$\mathcal{M}_n = \mathbb{C}(f_o,\dots,f_h) \; .$$

Hermitian modular functions, which were introduced by BRAUN [7] , represent another generalization of elliptic modular functions. Results by analogy with SIEGEL's theory are due to BRAUN [7,8] and KLINGEN [28,29].

All the quoted theories are special cases of a theory of automorphic forms and functions with respect to a discontinuously acting group with non-compact fundamental domain.

The contents of this volume

In this volume the theory of modular forms and functions on the half-space of quaternions is developed as another generalization of the classical theory of elliptic modular forms and functions. Under the objective of attaining to explicit results, it is mainly "SIEGEL's methods", which are used to derive analogous assertions to the case of SIEGEL's modular forms.

Let $\mathbb{H}$ denote the skew field of real quaternions and "-" its canonical involution. Thus the half-space of quaternions

(1") $$H(n;\mathbb{H}) := \left\{ Z = X + iY \in \mathrm{Mat}(n;\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} \, ; \; \begin{matrix} X=\overline{X}' \, , \, Y=\overline{Y}' \, , \\ Y \text{ positive definite} \end{matrix} \right\}$$

is defined as a subset of the tensor product $\mathrm{Mat}(n;\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$. In the case $n = 1$ one obtains the upper half in $\mathbb{C}$:

$$H(1;\mathbb{H}) = H(1;\mathbb{R}) = \mathcal{H} \; .$$

On the one hand SIEGEL's half-space $H(n;\mathbb{R})$ and the Hermitian half-space $H(n;\mathbb{C})$ can be considered as analytic submanifolds of $H(n;\mathbb{H})$. On the other hand it sometimes proves useful to imbed $H(n;\mathbb{H})$ into $H(2n;\mathbb{C})$ or $H(4n;\mathbb{R})$.

The symplectic group (2') is generalized to

(2") $\quad Sp(n;\mathbb{H}) = \{M \in Mat(2n;\mathbb{H})\ ;\ \bar{M}'JM = J\}$.

Hence all biholomorphic maps of $H(n;\mathbb{H})$ onto itself are given by symplectic transformations

(3") $\quad Z \longmapsto M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}\ ,\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n;\mathbb{H})$,

where in the case $n = 2$ one has to add the transposed map $Z \longmapsto Z'$.

The modular group is defined by

(4") $\quad \Gamma_n := Sp(n;\mathcal{O})$,

where $\mathcal{O}$ denotes the quaternions of HURWITZ [25]. Γ_n forms a discrete subgroup of $Sp(n;\mathbb{H})$ and acts discontinuously on $H(n;\mathbb{H})$. The validity of the Euclidean algorithm in $\mathcal{O}$ turns out to be the decisive factor to adopt results from SIEGEL's theory. The difficulties arising from the number theory, if one considers Hermitian modular forms, do not occur.

Now one defines $\det(CZ+D)^k$ for $Z \in H(n;\mathbb{H})$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n;\mathbb{H})$ and even $k \in \mathbb{Z}$ by means of the embedding of $H(n;\mathbb{H})$ into $H(2n;\mathbb{C})$. In virtue of a generalization of MINKOWSKI's reduction theory, one can proceed as SIEGEL [54] did in order to construct a fundamental domain.

<u>Theorem A".</u> $\mathcal{F}_n := \left\{ Z = X + iY \in H(n;\mathbb{H})\ ;\ \begin{array}{l} X \bmod 1\ ,\ Y\ \underline{\text{reduced}} \\ |\det(CZ+D)|^2 \geq 1\ \underline{\text{for}}\ M \in \Gamma_n \end{array} \right\}$

<u>is a fundamental domain of</u> $H(n;\mathbb{H})$ <u>with respect to the action of the modular group</u> Γ_n .

Given $n \geq 2$ and an even $k \in \mathbb{Z}$ a <u>modular form of quaternions of weight</u> k is defined by analogy to (5') and possesses a FOURIER-expansion corresponding to (6').

Classic procedure yields that the vector space $[\Gamma_n, k]$ of modular forms of quaternions having weight k proves finitely dimensional and consists only of 0 , whenever $k < 0$.

By applying PETERSSON's resp. MAASS' theory of metrization to modular forms of quaternions, theorems of representation are obtained. $[\Gamma_n, k]$ is spanned by generalized EISENSTEIN- and POINCARÉ -series, whenever $k > 8n - 6$ is even.

Theta-series are defined in correspondence to (8') and the theory of singular modular forms yields a basis of $[\Gamma_n, k]$ consisting of theta-series under the condition $0 < k < 2n$.

Summarizing one has

Theorem B". a) Every modular form of quaternions having negative weight vanishes identically and

$$\dim [\Gamma_n, k] < \infty$$

holds, whenever $k \geqq 0$ is even.

b) If $0 < k < 2n$ theta-series form a basis of $[\Gamma_n, k]$.

c) Let $k > 8n - 6$ be even, then $[\Gamma_n, k]$ is spanned by EISENSTEIN- and POINCARÉ -series.

If $n \geqq 2$ modular functions of quaternions are defined on the analogy of (9') to be meromorphic functions on $H(n;\mathbb{H})$, which remain invariant under all modular transformations. The set M_n of all modular functions of quaternions forms a field of transcendence degree $n(2n - 1)$. Modular functions again can be represented as quotients of modular forms having the same weight and one achieves

Theorem C". There exist modular functions of quaternions $f_o, \ldots, f_h$, $h = n(2n-1)$, such that

$$M_n = \mathbb{C}(f_o, \ldots, f_h) \ .$$

In view of the choice of the quaternions of HURWITZ as integral elements, it is possible to represent the theory of modular forms of quaternions, of Hermitian modular forms with respect to the Gaussian integers and of SIEGEL's modular forms throughout the whole volume simultaneously.

The method of compactification according to SATAKE [51] resp. BAILY and BOREL [4] will not be treated in the sequel. The theory of HECKE-operators is also left out of consideration, since the situation is completely different from that dealing with SIEGEL's modular forms. But an investigation of this subject will follow [41].

My thanks are due to Professor Dr. M. KOECHER for his encouragement and helpful advice.

Notations

Let $\mathbb{N}$ denote the natural numbers without 0 , $\mathbb{Z}$ the integral, $\mathbb{Q}$ the rational, $\mathbb{R}$ the real, $\mathbb{C}$ the complex numbers and $\mathbb{H}$ the skew field of real quaternions. Given a ring R and $m,n \in \mathbb{N}$ let $Mat(m,n;R)$ denote the set of matrices having m rows and n columns and coefficients in R . We use the abbreviations $Mat(n;R) := Mat(n,n;R)$ and $R^m := Mat(m,1;R)$. On the other hand $A^{(m,n)}$ also stands for a matrix having m rows and n columns as well as $A^{(n)}$ for $A^{(n,n)}$. If $A_1,\dots,A_n$ are quadratic matrices $[A_1,\dots,A_n]$ is defined to be that quadratic matrix, where $A_1,\dots,A_n$ stand on the diagonal and O's else. A' always denotes the transposed matrix of A . The letter $I = I^{(n)}$ is reserved for the identity matrix $[1,\dots,1]$, whereas 0 stands for a matrix of suitable size having only O's . The identical map of a non-empty set M onto itself is denoted by id_M . By γ_n we mean the symmetric group on n letters, i.e. the group of all bijective maps of $\{1,\dots,n\}$ onto itself. The KRONECKER-symbol is denoted by δ_{jk} and $ord\, M$ equals the number of elements of a set M.

Finally the way of citing is explained. The whole volume is divided into 6 chapters and each chapter into paragraphs. Throughout each paragraph the theorems, lemmata and propositions are numbered consecutively. Only if assertions of another chapter are quoted the chapter is cited by Roman numerals. The numbers in brackets refer to the bibliography at the end of this volume.