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Singular Ordinary Differential Operators and Pseudodifferential Equations

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To

Doris and Ulrike

INTRODUCTION

Various problems in physics and engineering lead to a linear ordinary differential equation whose coefficient of the highest derivative vanishes at certain points. Such an equation is called degenerate or singular, and a zero of the leading coefficient is said to be a singular point or singularity of the corresponding differential operator. In Chapters 1 and 2 of these notes we consider linear ordinary differential operators with a singular point at the origin:

$$A = x^q D^l + \sum_{0 \leq i < l} a_i(x) D^i \ , \ D = d/dx \ , \ q \in \mathbb{N} \ . \tag{1}$$

In case of the homogeneous equation $Ay = 0$ with analytic coefficients a_i, the investigation of singular differential equations has a long history. In particular, starting with the works of Fuchs and Poincaré in the second half of the last century, the asymptotic behaviour as $x \to 0$ of solutions to such equations has been studied by many authors; see e.g. Sternberg [1], Wasow [1], Ince [1]. In contrast to that, general results on the solvability of the degenerate inhomogeneous equation $Ay = f$ have been obtained only during the last fifteen years, using the methods of linear functional analysis.

In order to describe such a result and the material of these notes, it is necessary to recall some definitions. Let X and Y be linear topological spaces, and $A : X \to Y$ a linear operator with the domain of definition $D(A) \subset X$. If A is a continuous map of X into Y and $D(A) = X$, we shall write $A \in L(X,Y)$ and $A \in L(X)$ when $X = Y$. A is called normally solvable if its range $\operatorname{im} A = A(D(A))$ is closed in Y. The dimension $\dim \ker A$ of the kernel $\ker A = \{x \in D(A) : Ax = 0\}$ will be called the kernel index or nullity of A, and the deficiency $\dim Y/\operatorname{im} A$ of im A in Y will be called the deficiency index of A. If A is normally solvable and its kernel and deficiency indices are both finite, we say that A is a Fredholm operator, and its index is defined by $\operatorname{ind} A = \dim \ker A - \dim Y/\operatorname{im} A$. Furthermore, A is called Φ_+- (resp. Φ_--)operator if it is normally solvable and $\dim \ker A < \infty$ (resp. $\dim Y/\operatorname{im} A < \infty$). For Fredholm and $\Phi_\pm$-operators and their basic properties, we refer to the exposition in Goldberg [1], Prößdorf [1], Przeworska-Rolewicz and Rolewicz [1].

Malgrange [1], Komatsu [1] and Korobejnik [1] independently of each other found a general index formula for the operator (1) in the space $\mathcal{H}(\Omega)$ of analytic functions in a domain $\Omega \subset \mathbb{C}$, $0 \in \Omega$, which we state here in the special case of a simply connected domain:

If $a_i \in \mathcal{H}(\Omega)$ $(i=0,\dots,l-1)$, then $A \in L(\mathcal{H}(\Omega))$ is a Fredholm operator with index $l-q$.

Thus we observe that the index of A coincides with the index of the principal part $x^q D^l$ in $\mathcal{H}(\Omega)$. This result is not true, in general, if (1) acts in spaces of differentiable functions on an interval $[a,b] \subset \mathbb{R}$, $0 \in [a,b]$. Since the end of the sixties there appeared a lot of papers concerning the index and the solvability properties of the operator (1) in such spaces. The aim of Chapters 1 and 2 of this work is to fit most of those results into a general framework and thus to give a review of the current state of this field. Chapter 1 deals with the special case of a Fuchsian differential operator and also serves as an illustration of typical methods and results in the theory of singular ordinary differential equations. In Chapter 2 we give a general index formula for the differential operator (1) in the space of infinitely differentiable functions and in weighted L_p spaces on an interval. Furthermore, we study kernel, range, normal solvability in weighted Sobolev spaces and hypoellipticity of the degenerate operator (1) as well as its index in the space of distributions. A preliminary version of Chapters 1 and 2 appeared in Elschner and Silbermann [2].

In Chapter 3 these results will partly be generalized to differential operators with a finite number of singular points on compact or infinite intervals. In case of Fuchsian differential operators on a finite interval, the index formula is applied to study essential selfadjointness and spectrum of boundary value problems for those operators.

We remark that there is an increased interest in the development of the theory of singular ordinary differential operators in connection with the very active fields of solvability theory for degenerate partial differential equations (see e.g. Bolley, Camus and Helffer [1], [2], Baouendi and Goulaouic [1], Baouendi, Goulaouic and Lipkin [1], Helffer and Rodino [1], Višik and Grušin [1], Elschner and Lorenz [3], Lorenz [1]) and numerical analysis for degenerate ordinary differential equations (cf. e.g. de Hoog and Weiss [1], [2], Natterer [1], Elschner and Silbermann [1]).

Chapter 4 illustrates the application of singular ordinary differential operators in the theory of partial differential equations. Relying on certain results in Chapters 1 and 2, we study local solvability as well as normal solvability and index in Sobolev spaces for some examples of elliptic partial differential operators degenerating at one point. Many problems in this field are still open.

Various boundary value problems in mathematical physics and complex function theory lead to singular integro-differential, or more generally, to pseudodifferential equations on a closed curve. A pseudodifferential operator is called classical if its symbol has an asymptotic expansion as a sum of symbols which are positive homogeneous in the covariable of decreasing orders (cf. Chapter 5); it is called non-elliptic or degenerate if the principal term of the symbol vanishes at certain points on the cosphere bundle of the curve.

A major part of Chapter 5 is devoted to the author's recent results on the index and the Fredholm property of degenerate classical pseudodifferential operators on a closed contour, though, for the lack of space, we have not covered all of the material in full generality. As an application of these results, theorems on the index and on existence and uniqueness of solutions of the degenerate oblique derivative problem in the plane are given. Furthermore, in Chapter 5 we have included an almost self-contained introduction to classical pseudodifferential operators on a closed curve. For the general theory of pseudodifferential equations, the reader is referred to Šubin [1], Taylor [1] and Treves [1]. An exposition of the theory of degenerate one-dimensional singular integral equations which have been studied somewhat earlier can be found in Prößdorf [1] and Michlin and Prößdorf [1].

Chapter 6 deals with the Galerkin method using periodic splines as test and trial functions for the approximate solution of pseudodifferential equations on a closed contour. It demonstrates the interplay between certain a priori estimates for pseudodifferential operators, namely the Gårding and Melin inequalities, and convergence results for Galerkin's method with splines for strongly elliptic and degenerate equations. For an introduction to the theory of splines and finite element methods, we refer to Aubin [1], de Boor [1] and Strang and Fix [1]. The reader should consult the introduction and the section "comments and references" in each chapter for more information on the contents of these notes and further references.

Except for Chapter 4, the material is rather self-contained. The reader is assumed to be familiar with linear functional analysis (see e.g. Goldberg [1], Prößdorf [1]). In Chapter 4 some previous knowledge of elliptic differential operators on manifolds is desirable (cf. Narasimhan [1], Agranovič and Višik [1]).

Throughout the book the following notation is used. For a domain $\Omega \subset \mathbb{R}^n$, let $C^\infty(\Omega)$ $(C_0^\infty(\Omega))$ be the set of infinitely differentiable functions (with compact support) in Ω, and $C^\infty(\bar{\Omega})$ the set of all infinitely differentiable functions in Ω which together with all derivatives continuously extend to the closure $\bar{\Omega}$ of Ω. For $\Omega = (a,b) \subset \mathbb{R}$, we simply write $C^\infty(\Omega) = C^\infty(a,b)$, $C^\infty(\bar{\Omega}) = C^\infty[a,b]$ etc. The support of a function u is denoted by supp u. In $C^\infty(\bar{\Omega})$ (resp. $C_0^\infty(\Omega)$) one can introduce the topology of a Fréchet (resp. locally convex) space; see Hörmander [2], Robertson and Robertson [1]. The bilinear form

$$\langle u,v\rangle = \int_\Omega uv\, dx$$

on $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ extends to a duality between $C_0^\infty(\Omega)$ and the locally convex space $\mathcal{D}'(\Omega)$ of all distributions in Ω. For $A \in L(C_0^\infty(\Omega))$, the transpose ${}^tA \in L(\mathcal{D}'(\Omega))$ of A is defined by

$$\langle {}^tAu,v\rangle = \langle u,Av\rangle, \; u \in \mathcal{D}', \; v \in C_0^\infty.$$

Finally, if M is an n-dimensional infinitely differentiable manifold with or without boundary, let $C^\infty(M)$ be the set of all infinitely differentiable functions on M; see Narasimhan [1]. Other notation is either standard or defined upon introduction.

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