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Stratified Mappings – Structure and Triangulability

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INTRODUCTION

For several reasons, most of them stemming from algebraic topology, it is important to know whether a topological space, or more generally a continuous map, is triangulable or not. Cairns [Ca] proved the triangulability of smooth manifolds; another proof, also providing a uniqueness result, is due to J. H. C. Whitehead [Wh]. First attempts to prove the triangulability of algebraic sets are due van der Waerden [W], Lefschetz [Le], Koopman and Brown [K-B] and Lefschetz and Whitehead [L-W]. Rigorous proofs, in the more general case of semianalytic sets, were given by Lojasiewicz [Lo] and Giesecke [Gi]. Later, Hironaka [Hi_1] and Hardt [Ha_2] proved the triangulability of subanalytic sets. The most general spaces known to be triangulable are the stratified sets introduced by Thom [T_2] and the abstract stratifications introduced by Mather [Ma_1] (Mather's notion is slightly different from Thom's one, but it is more or less clear that the two classes of spaces coincide, at least in the compact case); they include all the spaces mentioned above and also the orbit spaces of smooth actions of compact Lie groups. Their triangulability was proved by several authors (Goresky [G_1], Johnson [J_2], Kato [Ka] and Verona [Ve_3] to mention only the published proofs. A more detailed discussion of these proofs and of others can be found in the introduction of Johnson's paper or at the end of Section 7 of the present work). The more difficult problem of the triangulability of mappings was considered by much fewer authors: Putz [P] proved the triangulability of smooth submersions, Hardt [Ha_2] proved the triangulability of some, very special, subanalytic maps and I proved in [Ve_3] the triangulability of certain stratified maps. In [T_1] Thom considered the problem of the triangulability of smooth maps and (implicitely) conjectured that "almost all" smooth mappings are triangulable. It is the aim of this paper to prove this conjecture. To be precise, we shall prove

Theorem. Let M and N be smooth manifolds without boundary. Then any proper, topologically stable smooth map from M to N is triangulable.

Since the set of proper and topologically stable smooth maps from M to N is dense in the set of all proper smooth maps from M to N (Thom-Mather theorem*) we obtain a positive answer to the above mentioned conjecture.

As a matter of fact, we prove a more general result concerning the triangulability of certain stratified mappings (Theorem 8.9) which implies the theorem stated above and also the following result (first proved by Hardt [Ha_2]): any proper light subanalytic map is triangulable (light means that the preimage of a discrete set is discrete). Our Theorem 8.9 is not as general as one would expect it. It applies only to proper and nice abstract Thom mappings (nice means that the mapping is finite to one when restricted to a certain subspace). It is natural to conjecture that the theorem is true for any proper abstract Thom mapping. A positive answer would solve another conjecture of Thom [T_1] : any proper Thom mapping (in Thom's terminology "application sans eclatement") is triangulable. The main difficulty in proving this more general version of Theorem 8.9 is explained in Section 8.15.3.

Since we are dealing with stratified spaces as introduced by Mather in [Ma_1] and since these lecture notes have never been published, I thought it would be useful to collect in a first part of the present work (Chapters 1, 2, and 3) the main results of the theory. Some of the proofs presented here are new and simpler than the original ones. For technical reasons, we are obliged to work with certain manifolds with corners, called here manifolds with faces. The necessary facts concerning them are presented in Chapter 4. In Chapter 5, we extend the theory of abstract strati-

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*) This result was conjectured by Thom [T_1]. In [T_3] and [T_2] he outlined a possible proof. Later Mather ([Ma_2] and [Ma_3]) filled in the details, made it rigorous and, by slightly changing some of Thom's concepts, cleaned up many technical points. Another proof (only slightly different) is presented in [Gib].

fications and abstract Thom mappings to the case when the strata are allowed to be manifolds with faces; most of the proofs are copies of the proofs presented in the first three chapters and so they are omitted. In Chapter 6, we prove some theorems concerning the structure of abstract stratifications and of abstract Thom mappings. In some sense, they can be viewed as a kind of "resolution of the singularities" in the C^∞-case. For example, Theorem 6.5 can be interpreted as saying that any abstract stratification of finite depth can be obtained from a manifold with faces by making certain identifications on the faces. Chapter 8 contains the main results of the paper (they were mentioned above). In an appendix, I have collected some facts from PL-topology which are needed in Chapters 7 and 8.

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