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Asymptotic Prime Divisors

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TO MARTHA

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Numerous people have participated in the study of asymptotic prime divisors, and I have tried to acknowledge, in the text, a sampling of their contributions. To do so entirely would be impossible, and I hope I have been fair in my selection. Certain people have been particularly helpful to me, as much in stimulating conversations as in specific results. I offer my gratitude to Paul Eakin, Ray Heitmann, Dan Katz and Keith Whittington. My special thanks goes, as it does so often, to Jack Ratliff.

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Nita Goldrick typed the manuscript. Her great skill and patience eased a difficult task.

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INTRODUCTION

Asymptotic prime divisors represent the interface of two major ideas in the study of commutative Noetherian rings. The first, the concept of prime divisors, is one of the most valued tools in the researcher's arsenal. The second is the fact that in a Noetherian ring, large powers of an ideal are well behaved, as shown by the Artin-Rees Lemma or the Hilbert polynomial.

Although its roots go back further, the recent interest in asymptotic prime divisors began with a question of Ratliff: What happens to $\mathrm{Ass}(R/I^n)$ as n gets large? He was able to answer a related question, showing that if $\overline{I}$ is the integral closure of I, then $\mathrm{Ass}(R/\overline{I^n})$ stabilizes for large n. In a later work, he also showed that $\mathrm{Ass}(R/\overline{I^n}) \subseteq \mathrm{Ass}(R/\overline{I^{n+1}})$. (Earlier, Rees had shown that if $P \in \mathrm{Ass}(R/\overline{I^n})$, some n, then $P \in \mathrm{Ass}(R/\overline{I^m})$ for infinitely many m.) Meanwhile, Brodmann answered the original question, proving that $\mathrm{Ass}(R/I^n)$ also stabilizes for large n. Since then, the topic of asymptotic prime divisors has been growing rapidly, the latest development being the advent of asymptotic sequences, a useful and interesting analogue of R-sequences.

These notes attempt to present the bulk of the present knowledge of asymptotic prime divisors in a reasonably efficient way, to ease the task of those wishing to learn of, or contribute to the subject. Modulo some gnashing of teeth, and rending of garments, it was both educational and satisfying to write them. I hope that reading them is the same.

The first chapter shows that for an ideal I in a Noetherian ring R, $\mathrm{Ass}(R/I^n)$ stabilizes for large n, as does $\mathrm{Ass}(I^{n-1}/I^n)$, the respective stable values of these two sequences are being denoted $A^*(I)$ and $B^*(I)$. Also $B^*(I)$ is characterized as the contraction to R of prime divisors Q of $t^{-1}\mathcal{R}$ with $It \not\subseteq Q$, where $\mathcal{R} = \mathcal{R}[t^{-1}, It]$ is the Rees ring of R with respect to I.

Chapter Two shows that $A^*(I) - B^*(I) \subseteq \mathrm{Ass}\, R$, and that $P \in A^*(I) - B^*(I)$ if and only if there is a $k \geq 1$ such that $P^{(k)}$ is part of a primary decomposition of I^n for all sufficiently large n.

Chapter Three shows that $\mathrm{Ass}(R/\overline{I}) \subseteq \mathrm{Ass}(R/\overline{I^2}) \subseteq \ldots$, and that this sequence eventually stabilizes to a set denoted $\bar{A}^*(I)$. Furthermore, $\bar{A}^*(I) \subseteq A^*(I)$. It also developes several technical results useful for dealing with $\bar{A}^*(I)$, the most important of these being that in a local ring, $P \in \bar{A}^*(I)$ if and only if there are primes $q^* \subseteq p^*$ in the completion R^* such that q^* is minimal, $p^* \cap R = P$ and $p^*/q^* \in \bar{A}^*(IR^* + q^*/q^*)$.

In Chapter Four, it is shown that if R is locally quasi-unmixed, then $P \in \bar{A}^*(I)$ if and only if height $P = \ell(I_P)$, the analytic spread of I_P. Since a complete local domain is locally quasi-unmixed, this result meshes nicely with the one mentioned from Chapter Three.

Chapter Five introduces asymptotic sequences: A sequence $x_1, \ldots, x_n$ such that $(x_1, \ldots, x_n) \neq R$ and for $i = 0, \ldots, n-1$, $x_{i+1} \notin \cup \{P \in \bar{A}^*((x_1, \ldots, x_i))\}$. In a local ring (R,M) it is shown that $x_1, \ldots, x_n$ is an asymptotic sequence if and only if $\mathrm{height}((x_1, \ldots, x_n)R^* + q^*/q^*) = n$ for each minimal prime q^* of the completion. This is then used to show that for a given ideal I in any Noetherian ring, all asymptotic sequences maximal with respect to coming from I have the same length, denoted $\mathrm{gr}^* I$. It is then shown that asymptotic sequences are to locally quasi-unmixed rings as R-sequences are to Cohen-Macaulay rings.

In Chapter Six, the sequence $x_1, \ldots, x_n$ is called an asymptotic sequence over the ideal I if $(I, x_1, \ldots, x_n) \neq R$ and for $i = 0, \ldots, n-1$, $x_{i+1} \notin \cup \{P \in \bar{A}^*((I, x_1, \ldots, x_i))\}$. It is shown that in a local ring, all maximal asymptotic sequences over I have the same length.

Chapter Seven proves that in a local ring, the grade of R/I^n stabilizes for large n, and gives partial results concerning $\mathrm{gr}(R/\overline{I^n})$.

Chapter Eight identifies, with one possible exception, all Noetherian rings for which $A^*(I) = \bar{A}^*(I)$ for all ideals I.

In Chapter Nine, asymptotic prime divisors play a minor role in proving the following unexpected result. Let P be prime in a Noetherian domain. Then there is a chain of ideals $P = I_0 \subset I_1 \subset \ldots \subset I_n$ with the following property: Let Q be a prime containing P, and let j be the largest subscript such that $I_j \subseteq Q$. Then $P \subseteq Q$ satisfies going down if and only if j is even.

In Chapter Ten, we consider a local ring (R,M) and the ideal transform of M, $T(M)$. Previously it was known that the following two statements are equivalent: (a) $T(M)$ is an infinite R-module (b) The completion of R contains a depth 1 prime divisor of zero. Our main result adds two more equivalent conditions: (c) $M \in A^*(J)$ for every regular ideal J (d) There is a regular element x with $M \in A^*(J)$ for all $J \sim xR$. Here $J \sim I$ if for some n and m, I^n and J^m have the same integral closure. Motivated by statement (d), we then discuss the possibility of defining a strong asymptotic sequence $x_1, \ldots, x_n$ with $(x_1, \ldots, x_n) \neq R$ and for $i = 0, \ldots, n-1$ $x_{i+1} \notin \cup\{P \in \cap A^*(J) | J \sim (x_1, \ldots, x_i)\}$, in the hope that such a sequence will stand in relation to prime divisors of zero, as asymptotic sequences stand to minimal primes. This program is carried out for $n = 1$ and 2.

Chapter Eleven is aptly titled Miscellaneous. It contains topics (of varying worth) which did not fit elsewhere.

The study of asymptotic prime divisors frequently impinges on that of the structure of the spectrum of a Noetherian ring, often referred to as the study of chain conditions. I have tried to keep to a minimum the amount of knowledge of chain conditions necessary to read these notes. In the Appendix, I list those definitions and basic results (with references for the curious reader) which are referred to in the text.