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Diophantine Approximation

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Preface

In spring 1970 I gave a course in Diophantine Approximation at the University of Colorado, which culminated in simultaneous approximation to algebraic numbers. A limited supply of mimeographed Lecture Notes was soon gone. The completion of these new Notes was greatly delayed by my decision to add further material.

The present chapter on simultaneous approximations to algebraic numbers is much more general than the one in the original Notes. This generality is necessary to supply a basis for the subsequent chapter on norm form equations. There is a new last chapter on approximation by algebraic numbers. I wish to thank all those, in particular Professor C.L. Siegel, who have pointed out a number of mistakes in the original Notes. I hope that not too many new mistakes have crept into these new Notes.

The present Notes contain only a small part of the theory of Diophantine Approximation. The main emphasis is on approximation to algebraic numbers. But even here not everything is included. I follow the approach which was initiated by Thue in 1908, and further developed by Siegel and by Roth, but I do not include the effective results due to Baker. Not included is approximation in p-adic fields, for which see e.g. Schlickewei [1976, 1977], or approximation in power series fields, for which see e.g., Osgood [1977] and Ratliff [1978]. Totally missing are Pisot-Vijayaraghavan Numbers, inhomogeneous approximation and uniform distribution. For these see e.g. Cassels [1957] and Kuipers and Niederreiter [1974]. Also excluded are Weyl Sums, nonlinear approximation and diophantine inequalities involving forms in many variables.

My pace is in general very leisurely and slow. This will be especially apparent when comparing Baker's [1975] chapter on approximation to algebraic numbers with my two separate chapters, one dealing with Roth's Theorem on approximation to a single algebraic number, the other with simultaneous approximation to algebraic numbers.

Possible sequences are chapters

I, II, III, for a reader who is interested in game and measure theoretic results, or

I, II, V, for a reader who wants to study Roth's Theorem , or

I, II, IV, V, VI, VII (§ 11, 12), VIII (§ 7-10), for a general theory of simultaneous approximation to algebraic numbers, or

I, II, IV, V, VI, VII, if the goal is norm form equations, or

I, II, VIII (§ 1-6, §11), if the emphasis is on approximation by algebraic numbers.

December 1979 W.M. Schmidt

Notation

A real number ξ may uniquely be written as

$$\xi = [\xi] + \{\xi\} ,$$

where $[\xi]$, the <u>integer part of</u> ξ, is an integer, and where $\{\xi\}$, the <u>fractional part of</u> ξ, satisfies $0 \leqq \{\xi\} < 1$.

$\|\xi\| = \min(\{\xi\}, 1 - \{\xi\})$ is the distance from ξ to the nearest integer,

U denotes the unit interval $0 \leqq \xi < 1$.

$\mathbb{R}^n$ denotes the n-dimensional real space,

E^n denotes Euclidean n-space.

$\underline{\underline{x}}, \underline{\underline{y}}, \ldots$ will denote vectors; so $\underline{\underline{x}} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, or

$\underline{\underline{x}} = (x_1, \ldots, x_m) \in \mathbb{R}^m$, etc.

Addition and multiplication of vectors by scalars is obvious.

$\underline{\underline{e}}_1, \ldots, \underline{\underline{e}}_n$ will denote basis vectors.

λK, where $\lambda > 0$ and where K is in $\mathbb{R}^n$, is the set of elements $\lambda\underline{\underline{x}}$ with $\underline{\underline{x}} \in K$.

δ_{ij} is the Kronecker Symbol.

$X, Y, \ldots$, in general will be variables, while $x, y, \ldots$ will be real, usually rational integers. But this rule is sometimes hard to follow: In chapter IV, the symbols $X, Y, \ldots$ will also be used to denote coordinates in compound spaces.

$\overline{|\underline{\underline{x}}|} = \max(|x_1|, \ldots, |x_n|)$ if $\underline{\underline{x}} = (x_1, \ldots, x_n)$. However

$\overline{|\underline{\underline{\beta}}|}$, where $\underline{\underline{\beta}} = (\beta_1, \ldots, \beta_n)$ has coordinates in an algebraic number field K, is given by $\overline{|\underline{\underline{\beta}}|} = \max(|\beta_1^{(1)}|, \ldots, |\beta_n^{(1)}|, \ldots, |\beta_1^{(k)}|, \ldots, |\beta_n^{(k)}|)$, if $\beta^{(1)} = \beta$, $\beta^{(2)}, \ldots, \beta^{(k)}$ are the conjugates of an elements β

(But, on p. 173, $\overline{|\gamma|}$ for a single element γ has a different meaning.)

$\overline{|P|}$ is the maximum absolute value of the coefficients of a polynomial P ,

$\mathbb{Q}$ is the field of rationals,

$\mathbb{R}$ is the field of reals,

$\mathbb{C}$ is the field of complex numbers.

$[L : K]$ is the degree of a field extension L over K .

$\{a,b,\dots,w\}$ denotes the set consisting of $a,b,\dots,w$, and

$\sim$ denotes a set theoretic difference.

$\ll$ is the Vinogradov symbol. Thus e.g. $f(\underline{\underline{x}}) \ll g(\underline{\underline{x}})$ means that $|f(\underline{\underline{x}})| \leq c|g(\underline{\underline{x}})|$ with a constant c . Often this "implied" constant c may depend on extra parameters, such as the dimension, etc.

$\gg\ll$, in the context $f \ll g$, means that both $f \ll g$ and $g \ll f$.

o , the "little o" , in the context $f(n) = o(g(n))$, means that $f(n)/g(n)$ tends to 0 as $n \to \infty$.

g.c.d. denotes the greatest common divisor of integers.

Starred Theorems, such as Theorem 6A* , are not proved in these Notes.

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