Lecture Notes in Mathematics

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Preface

These lecture notes are based on a series of lectures I gave at the Institute for Advanced Study in 1971. The lectures were written up by John Hinrichsen, and I am very grateful to him for undertaking the task. I am also indebted to George Wilson who helped me revise and improve them.

CONTENTS

Introduction

These lectures, based on joint work with I. M. Singer, will describe an extension of the index theory of elliptic operators beyond that developed in [7], [8], [9]. In those papers we studied an elliptic operator P invariant under a compact Lie group G. Its index is a character of G defined by

$$\text{index } P = \text{character } (\text{Ker } P) - \text{character } (\text{Ker } P^*)$$

and a general formula for it was obtained in terms of the geometrical data. In fact, by density arguments, this is really a result about finite groups and G enters essentially in an algebraic way. In the present lectures we shall consider a more general situation in which G enters analytically. Roughly speaking we shall produce a synthesis of the index theory of elliptic operators with the Fourier analysis of compact groups.

Given an action of G on a compact manifold X we shall consider a differential (or pseudo-differential) operator P on X which is G-invariant and is elliptic in the directions transversal to the orbits of G. For such an operator, Ker P may be infinite-dimensional, but each irreducible representation of G will only occur in Ker P with finite multiplicity. Moreover, these multiplicities do not grow too fast with the representation so that the character of Ker P is well-defined as a distribution on G. The same being true for P^* we can then also define index P as a distribution on G. Our aim is to give a method of computing this index-distribution in topological terms, starting from the symbol of P.

If G is finite then transversal ellipticity coincides with ellipticity and we are back in the earlier situation. On the other hand if X is a homogeneous space G/H every operator (including zero) is transversally elliptic and we have only to use Fourier analysis on G: the index formula becomes essentially the Frobenius reciprocity formula for induced representations. The general case is a mixture of the two in which group invariance along the orbits gives the control that is missing in the partial ellipticity of P.

A transversally elliptic operator on a free G-manifold X can be reduced to an elliptic operator on the orbit space X/G. When the action is not free, this reduction is not possible, but a transversally elliptic operator may still be interpreted as some kind of operator on the singular space X/G. As one of our applications, we apply this observation to the case where G acts on X with finite isotropy groups. Then X/G is a rational homology manifold and a cohomological formula is obtained for its Hirzebruch signature.

Lectures 1-4 are fairly straightforward generalizations of the results in [8]. The first novel point is the proof in lecture 2 that the distribution-index is well defined. The multiplicative property (lecture 3) is in fact somewhat simpler to formulate in the present more general framework. We should also draw attention to the general process of inducing (lecture 4) which enables us always to enlarge the group G. Finally an important step is the argument in lecture 4 which enables us to reduce from a connected group to its maximal torus T by using the complex structure on the flag manifold G/T.

Lectures 5-8 are entirely concerned with the special case when G is a torus acting linearly on a Euclidean space (or rather on its compactification, the sphere). This is really new because the corresponding problem in [8] for fully elliptic operators is an immediate consequence of the periodicity theorem. Lectures 5 and 6 treat the case when G is a circle and lectures 7 and 8 then extend this by induction to the torus. Lecture 7 involves some rather sophisticated commutative algebra - the theory of Cohen - Macaulay rings. Lecture 9 then translates the results into equivariant cohomology (the analogue of [10]) and lecture 10 gives some applications in the case of finite isotropy groups.

Although the problem of computing the distribution-index in terms of its symbol is completely solved in principle in these lectures, the solution falls short of providing an explicit general formula. For a torus acting with only finite isotropy groups the results in Lecture 9 do provide satisfactory formulae, and for a circle (with any action) the results are also quite explicit. However for the general case we give only a reduction process and one might hope for something more explicit. This probably requires the development of an appropriate algebraic machinery, involving cohomology but going beyond it.