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Jordan Pairs

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INTRODUCTION

The theory of Jordan algebras, compared to that of associative algebras, presents some unusual features. Recall that a unital quadratic Jordan algebra (in the sense of McCrimmon) is a triple $(J,U,1)$ consisting of a module J over some commutative associative ring k of scalars, a quadratic map U from J into the endomorphism ring $\mathrm{End}_k(J)$ of J, and an element $1 \in J$ (the unit element) such that the following identities hold in all scalar extensions.

(1) $U_1 = \mathrm{Id}$,

(2) $\{x,y,U_x z\} = U_x\{yxz\}$,

(3) $U(U_x y) = U_x U_y U_x$.

Here $\{xyz\} = U_{x+z}y - U_x y - U_z y$ is the linearization of $U_x y$. The standard example is an associative algebra with $U_x y = xyx$. Thus for one thing, Jordan algebras are not algebras in the usual sense since they are not based on a bilinear multiplication but rather on the composition $U_x y$ which is quadratic in x and linear in y. More serious is the important role played by the notion of isotope. Let $v \in J$ be invertible with inverse $v^{-1} = U_v^{-1}.v$, and set $U_x^{(v)} = U_x U_v$, for all $x \in J$. Then $(J,U^{(v)},v^{-1})$ is a unital quadratic Jordan algebra, called the v-isotope of J, and denoted by $J^{(v)}$. If v is not invertible then one can still define a non-unital Jordan algebra $J^{(v)}$, the v-homotope of J. We will call two Jordan algebras J and J' isotopic if there exists an isotopism between them, i.e., an isomorphism from J onto some isotope of J'. Several important theorems in the theory of Jordan algebras hold only up to isotopy; i.e., they are of the form " J may not have property (P) but there exists an isotope of J which does". Closely related to this is the fact that the autotopism group of J, usually called the structure group and denoted by $Str(J)$, plays a more important role than the automorphism group. For example, there is no natural concept of inner automorphism for Jordan algebras (which would be comparable to the inner automorphisms $x \mapsto axa^{-1}$ in an associative algebra) but every invertible element $a \in J$ defines an inner autotopism, namely U_a. All this suggests that there ought to be some algebraic object associated with J which somehow incorporates all homotopes of J and whose automorphism group is just the structure group of J. This object is the Jordan pair (J,J) associated with J.

Let us now describe this concept. A Jordan pair is a pair $V = (V^+,V^-)$ of k-modules together with quadratic maps $Q_+: V^+ \to Hom_k(V^-,V^+)$ and $Q_-: V^- \to Hom_k(V^+,V^-)$ which satisfy the following identities in all scalar extensions.

JP1 $\{x,y,Q_\sigma(x)z\} = Q_\sigma(x)\{yxz\}$,

JP2 $\{Q_\sigma(x)y,y,z\} = \{x,Q_{-\sigma}(y)x,z\}$,

JP3 $Q_\sigma(Q_\sigma(x)y) = Q_\sigma(x)Q_{-\sigma}(y)Q_\sigma(x)$.

Here $\{xyz\} = Q_\sigma(x+z)y - Q_\sigma(x)y - Q_\sigma(z)y$ is, similarly as before, the linearization of $Q_\sigma(x)y$, and the index σ takes the values $+$ and $-$. A standard example is $V^+ = M_{p,q}(R)$, $V^- = M_{q,p}(R)$, rectangular matrices with coefficients in an associative algebra R, with $Q_\sigma(x)y = xyx$. By a homomorphism $h: V \to W$ between Jordan pairs we mean a pair $h = (h_+,h_-)$ of linear maps, $h_\sigma: V^\sigma \to W^\sigma$, such that $h_\sigma(Q_\sigma(x)y) = Q_\sigma(h_\sigma(x))h_{-\sigma}(y)$. From the well-known identity $\{U_x y,y,z\} = \{x,U_y x,z\}$ which holds in any Jordan algebra it is clear that we obtain a Jordan pair from a Jordan algebra J by setting $V^+ = V^- = J$ and $Q_+ = Q_- = U$. This Jordan pair will be denoted by (J,J). Also, if $g \in Str(J)$ and $g^\# \in Str(J)$ is defined by $g^\# = g^{-1}U_{g(1)}$ then $(g,(g^\#)^{-1})$ is an automorphism of the Jordan pair (J,J), and it turns out that the map $g \mapsto (g,(g^\#)^{-1})$ is an isomorphism between $Str(\mathcal{J})$ and the automorphism group of $(\mathcal{J},\mathcal{J})$. Theorems which for Jordan algebras only hold up to isotopy will then hold for the associated Jordan pairs without this restriction.

In an arbitrary Jordan pair $V = (V^+, V^-)$ we still have the concept of homotope as follows. For every $v \in V^-$ the module V^+ becomes a (in general non-unital) Jordan algebra, denoted by V^+_v, with quadratic operators $U_x = Q_+(x)Q_-(v)$ and squaring operation $x^2 = Q_+(x)v$. Thus the space which parametrizes the homotopes (namely V^-) is different from the space in which the homotope lives (namely V^+). By interchanging the roles of V^+ and V^- we can also define a homotope V^-_u for every $u \in V^+$. The condition that a Jordan pair V be of the form $(\mathcal{J},\mathcal{J})$, where $\mathcal{J}$ is a unital Jordan algebra, is now that V^- contains an invertible element; i.e., an element v such that $Q_-(v)$ is invertible. In this case, $\mathcal{J} = V^+_v$ is a unital Jordan algebra with unit element $Q_-(v)^{-1}v$, and V is isomorphic as a Jordan pair with $(\mathcal{J},\mathcal{J})$. Roughly speaking, therefore, Jordan pairs containing invertible elements are the same as unital Jordan algebras "up to isotopy".

In general, however, a Jordan pair will not contain any invertible elements. To see what is happening in this case, let us first make some remarks on Jordan algebras without unit element. There are two approaches to this: either a non-unital Jordan algebra $\mathcal{J}$ is defined in terms of quadratic operators U_x and a squaring operation x^2 (as for example V^+_v above), or one dispenses with the squaring altogether and retains only the quadratic operators. The first approach leads right back to unital Jordan algebras since $\mathcal{J}$ can be imbedded into a unital Jordan algebra $k.1 + \mathcal{J}$ by adjoining a unit element 1. The second approach leads to the concept of Jordan triple system, defined as a k-module $\mathcal{T}$ together with a quadratic map $U: \mathcal{T} \to End_k(\mathcal{T})$ satisfying the following identities in all scalar extensions.

JT1 $\{x,y,U_x z\} = U_x\{yxz\}$,

JT2 $\{U_x y,y,z\} = \{x,U_y x,z\}$,

JT3 $U(U_x y) = U_x U_y U_x$.

(The terminology "triple system" is due to the fact that in case 2 is invertible in k the theory can be based on the trilinear composition $\{xyz\} = U_{x+z}y - U_x y - U_z y$). If we compare these identities with those for a Jordan pair then it is obvious that T gives rise to a Jordan pair (T,T) by setting $V^+ = V^- = T$ and $Q_+ = Q_- = U$. Not every Jordan pair is of this form, however, since it is easy to construct examples of Jordan pairs for which V^+ and V^- are not isomorphic as k-modules. To obtain a Jordan triple system from a Jordan pair, we must have some way of identifying V^+ and V^- . More precisely, we define an involution of a Jordan pair to be a module isomorphism $\alpha\colon V^+ \to V^-$ such that $Q_-(\alpha(x)) = \alpha Q_+(x)\alpha$ for all $x \in V^+$. Then a Jordan pair with involution gives rise to a Jordan triple system by setting $T = V^+$ and $U_x y = Q_+(x)\alpha(y)$, and this establishes a one-to-one correspondence between Jordan triple systems and Jordan pairs with involution.

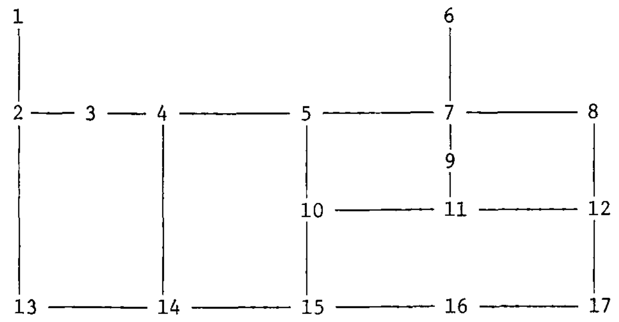
So far we have shown that Jordan pairs provide a unifying framework for both the theory of Jordan algebras and Jordan triple systems. Let us now point out some of the advantages which the the Jordan pair concept offers over both these theories.

In contrast to the case of Jordan algebras and triple systems, there is a natural way of defining inner automorphisms of Jordan pairs. Let $V = (V^+,V^-)$ be a Jordan pair, and consider a pair (x,y) where $x \in V^+$ and $y \in V^-$ (for which we simply write $(x,y) \in V$). We say that (x,y) is quasi-invertible if x is quasi-invertible in the Jordan algebra V^+_y ; i.e., if $1 - x$ is invertible in the Jordan algebra obtained from V^+_y by adjoining a unit element. In this case, (x,y) defines an inner automorphism $\beta(x,y)$ (cf. 3.9). Thus the quasi-invertible pairs are analogous to the invertible elements in an associative algebra. It is irrelevant for this whether V contains invertible elements or not; in fact, for most of the theory of Jordan pairs there is no difference between the two cases. These inner automorphisms play an important role and can be used to give a computation-free treatment of the Peirce decomposition (§5).

Another reason why Jordan pairs are preferable to Jordan algebras or triple systems is that they always contain sufficiently many idempotents. It may happen even in a finite-dimensional simple Jordan algebra that the unit element cannot be written as the sum of orthogonal division idempotents (the algebra need not have "capacity"). The situation is even worse for Jordan triple systems. Here an idempotent is an element x such that $U_x x = x$. In general, there are no such elements except zero; e.g., consider the real numbers with $U_x y = -x^2 y$. If V is a Jordan pair we define an idempotent to be a pair $(x,y) \in V$ such that $Q_+(x)y = x$ and $Q_-(y)x = y$. Then it turns out that a Jordan pair with dcc on principal inner ideals which is not radical always contains non-zero idempotents (§10). Of course, pairs (x,y) with $U_x y = x$ and $U_y x = y$ have been considered before in the theory of Jordan algebras but their natural place seems to be in the context of Jordan pairs. The scarcity of idempotents in the Jordan triple case is also explained. Indeed, under the correspondence between Jordan triple systems and Jordan pairs with involution, idempotents of the Jordan triple system correspond to idempotents (x,y) of the Jordan pair which are invariant under the involution in the sense that $y = \alpha(x)$, and there may be none of these.

Finally, let us mention that Jordan pairs arise naturally in the Koecher-Tits construction of Lie algebras and the associated algebraic groups, a topic not touched upon in these notes. Indeed, it was in this context that they were first introduced by K. Meyberg. For details, we refer to a forthcoming paper (Loos[7]).

We give now a more detailed description of the contents of these notes. There are 17 sections whose logical interdependence is summarized in the following diagram.



Here j depends on i if it stands below and/or to the right of i .

Chapter I (§§1 - 5) contains the general theory of Jordan pairs, beginning with their relationship to Jordan algebras and triple systems as discussed above. Just as in case of Jordan algebras, a long list of identities is required, and this is derived in §2. After the quasi-inverse (§3) various radicals are discussed in §4. The Jacobson radical of a Jordan pair V is directly based on the quasi-inverse, being defined by $\mathrm{Rad}\, V = (\mathrm{Rad}\, V^+, \mathrm{Rad}\, V^-)$ where $\mathrm{Rad}\, V^\sigma$ is the set of all properly quasi-invertible elements of V^σ (cf. 4.1). In §5, we introduce the Peirce decomposition

$$V = V_2(e) \oplus V_1(e) \oplus V_0(e)$$

of a Jordan pair with respect to an idempotent $e = (e^+, e^-)$ (5.4). Note that we use the indices 2,1,0 instead of the traditional 1,1/2,0 to label the Peirce spaces. Each Peirce space $V_i(e) = (V_i^+, V_i^-)$ is a subpair of V , and we have $\mathrm{Rad}\, V_i(e) = V_i(e) \cap \mathrm{Rad}\, V$ (5.8). There is also a Peirce decomposition with respect to an orthogonal system of idempotents (5.14).

Chapter II (§§6 - 9) is devoted to alternative pairs. An alternative pair is a pair $A = (A^+, A^-)$ of k-modules together with trilinear maps $A^+ \times A^- \times A^+ \to A^+$ and $A^- \times A^+ \times A^- \to A^-$, written $(x,y,z) \mapsto \langle xyz\rangle$, which satisfy the identities

AP1 $\quad \langle uv\langle xyz\rangle\rangle + \langle xy\langle uvz\rangle\rangle = \langle\langle uvx\rangle yz\rangle + \langle x\langle vuy\rangle z\rangle$,

AP2 $\quad \langle uv\langle xyx\rangle\rangle = \langle\langle uvx\rangle yx\rangle$,

AP3 $\quad \langle xy\langle xyz\rangle\rangle = \langle\langle xyx\rangle yz\rangle$.

In analogy with the Jordan case, alternative pairs containing invertible elements correspond to isotopism classes of unital alternative algebras, and alternative pairs with involution correspond to alternative triple systems. In contrast to the situation for alternative algebras, there exist simple properly alternative pairs of arbitrary (even infinite) dimension over their centroids. They can be constructed from alternating bilinear forms (6.6). Just as an alternative algebra gives rise to a Jordan algebra by setting $U_x y = xyx$ so we obtain a Jordan pair A^J from an alternative pair A by setting $Q_\pm(x)y = \langle xyx\rangle$. This relation is exploited in §7 to prove results about alternative pairs by passing to the associated Jordan pair. In §9, we study the Peirce decomposition of alternative pairs which is the tool for their classification in §11.

The main reason why alternative pairs are of interest to us, however, is that they arise naturally in the study of Jordan pairs without invertible elements. To explain this connection, let e be an idempotent of a Jordan pair V with the property that $V_0(e) = 0$. Then $V_1(e)$ becomes an alternative pair by setting $\langle xyz\rangle = \{\{xye^\sigma\}e^{-\sigma}z\}$ (8.2). Conversely, every alternative pair can be obtained in this way by means of the standard imbedding (8.12). Consider now a simple and semisimple Jordan pair with acc and dcc on principal inner ideals. Then we can always find an idempotent e with $V_0(e) = 0$. If $V_1(e)$ is also zero then $V = V_2(e)$ contains invertible elements and is therefore essentially a unital Jordan algebra up to isotopy. In view of the work of N. Jacobson and K. McCrimmon, this case may be considered as well known. If, on the other hand, $V_1(e) \neq 0$ then V is isomorphic with the standard imbedding of $V_1(e)$ (12.5).

In Chapter III (§§10 - 12) we present the structure theory of alternative and Jordan pairs with chain conditions on principal inner ideals. Inner ideals are introduced in §10. The theory follows the one for Jordan algebras but is actually simpler since the minimal inner ideals of type II have no analogue for Jordan pairs (10.5). In §11, we classify simple alternative pairs A containing an idempotent e with $A_{00}(e) = 0$ (11.11), and also under various chain conditions (11.16, 11.18). Finally, §12 contains the classification of semisimple Jordan pairs with dcc and acc on principal inner ideals (12.12), based on the connection with alternative pairs as explained above.

In Chapter IV (§§13 - 17) we consider finite-dimensional Jordan pairs over a field. After introducing universal enveloping algebras in §13 (which properly belongs to §2) the main result of §14 is that the radical of a finite-dimensional Jordan pair is nilpotent (14.11). It is an outstanding problem to extend this result to Jordan pairs with chain conditions on inner ideals. Unfortunately, there seems to be little hope to generalize the present proof since it uses the finite-dimensionality of the universal envelope and Engel's theorem (14.9). In § 15, we study Cartan subpairs of Jordan pairs. They are defined as associator nilpotent subpairs which are equal to their own normalizers. Using techniques similar to those in the theory of Cartan subgroups of algebraic groups, we show that any finite-dimensional Jordan pair contains Cartan subpairs (15.20), and that any two Cartan subpairs are conjugate by an inner automorphism, provided the base field is algebraically closed (15.17). The proofs depend on the fact that the orbit of a Cartan subpair under the inner automorphism group is dense in the Zariski topology (15.15). This also allows us to compute the generic minimum polynomial of a Jordan pair by its restriction to a Cartan subpair (16.15). The generic minimum polynomial is defined as the exact denominator of a suitable rational map (essentially the quasi-inverse, cf. 16.2). In contrast to the case of Jordan algebras, the degree of the generic minimum polynomial of a Jordan pair V does in general not coincide with the degree of V. This is the case, however, if V contains invertible elements or is separable. The generic trace of a separable Jordan pair may be degenerate in characteristic two, a phenomenon familiar from the theory of quadratic Jordan algebras. The generic norm, however, defined as the exact denominator of the quasi-inverse, is always non-degenerate in a certain sense (16.13). Finally, in §17, we work out the classification of simple finite-dimensional Jordan pairs over algebraically closed fields, using the results of §12. It turns out that such a Jordan pair is uniquely determined by three numerical invariants, dimension, rank, and genus, and also that the classification is independent of the characteristic of the base field (17.12).

In the Notes at the end of each chapter I have tried to make some historical comments, give credit where it is due, and also point out some open problems. I apologize in advance for any omissions or inaccuracies. In order to keep the text at a reasonable length, I have assumed as known the theory of quadratic Jordan algebras, to the extent of N. Jacobson's Tata Lecture Notes. In particular, the classification of semisimple unital Jordan algebras with dcc on principal inner ideals is not reproduced here from the Jordan pair point of view.

Most of the material was presented in a seminar at the University of British Columbia during the academic year 1973/74, and I wish to thank C.T. Anderson and M. Slater for their patience as my audience, and for many valuable remarks and suggestions. I am also indebted to E. Goodaire for proofreading the manuscript.

Vancouver, Summer 1974

O. Loos

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NOTATIONS AND CONVENTIONS

0.1. $\underline{\underline{Z}}$ is the ring of integers and $\underline{\underline{N}}$ denotes the set of non-negative integers. The unspecified term "ring" or "algebra" always means an associative (but not necessarily unital or commutative) ring or algebra.The commutator of two elements a and b of a ring is denoted by $[a,b] = ab - ba$. Fields are always commutative. Jordan algebras are always quadratic (but not necessarily unital) Jordan algebras in the sense of McCrimmon.

0.2. Throughout, k denotes a commutative unital ring of scalars. By an extension of k we mean a commutative unital k-algebra. If K is an extension of k then the natural homomorphism $k \to K$, $a \mapsto a.1$, need not be injective (e.g., $k = \underline{\underline{Z}}$ and $K = \underline{\underline{Z}}/p\underline{\underline{Z}}$). An extension field of k is an extension which is a field. The category of commutative unital k-algebras is denoted by k-$\underline{\underline{alg}}$. The symbol T usually stands for an indeterminate. Thus $k[T]$ is the polynomial algebra in one variable over k . The truncated polynomial rings $k[T]/(T^n)$ are denoted by $k(\varepsilon)$, $\varepsilon^n = 0$. For $n = 2$ this is the algebra of dual numbers over k .

0.3. All k-modules are unital. The symbol $\otimes$ stands for $\otimes_k$. If V is a k-module and R is an extension of k then the R-module

$$V_R = V \otimes R$$

is called the module obtained from V by extending the scalars to R , or simply a scalar extension of V . The image of an element x of V under the map $x \mapsto x \otimes 1$ from V into V_R is also denoted by x_R or even simply by x , although this map is in general not injective.

Let $k' \to k$ be a homomorphism of commutative rings with unity. Then the k-module V , considered as a module over k' , is called the module obtained from V by restricting the scalars to k' , and is denoted by ${}_{k'}V$. In particular, ${}_{\underline{\underline{Z}}}V$ is just the underlying abelian group of V .

If V and W are k-modules then $\mathrm{Hom}_k(V,W)$ or simply $\mathrm{Hom}(V,W)$ is the k-module of k-linear maps from V to W . Also, $\mathrm{End}(V) = \mathrm{Hom}(V,V)$ is the k-algebra of k-linear endomorphisms of V , and $GL(V)$ is the group of invertible elements of $\mathrm{End}(V)$.

0.4. A map $Q: V \to W$ between k-modules is called quadratic if $Q(\lambda x) = \lambda^2 Q(x)$ for all $\lambda \in k$, and if

$$Q(x,y) = Q(x+y) - Q(x) - Q(y)$$

is a bilinear map from $V \times V$ into W . For every extension R of k there is a unique quadratic map $Q_R: V_R \to W_R$ such that $Q_R(x_R) = Q(x)_R$ (see Jacobson[3] for a proof). Usually, we simply write Q instead of Q_R . A quadratic map q from V into k is called a quadratic form. We say q is non-degenerate if $q(x) = q(x,y) = 0$ for all y implies $x = 0$.

0.5. Let R be a non-associative (i.e., not necessarily associative) k-algebra We denote by R^{op} the opposite algebra, having the same underlying k-module and multiplication $a \cdot b = ba$. The identity maps $R \to R^{op}$ and $R^{op} \to R$ are antiisomorphisms, usually written $a \mapsto \bar{a}$.

The set of $p \times q$ matrices with entries in R is denoted by $M_{p,q}(R)$.

Instead of $M_{p,p}(R)$ we simply write $M_p(R)$. The transpose of a matrix x in $M_{p,q}(R)$ is denoted by ${}^t x$; it belongs to $M_{q,p}(R)$. We denote by $x^* = {}^t\bar{x}$ the transpose with coefficients in R^{op} so that $x^* \in M_{q,p}(R^{op})$. Then we have

$$(x^*)^* = x \quad \text{and} \quad (xy)^* = y^* x^* .$$

(Note that ${}^t(xy) \neq {}^t y\, {}^t x$ unless R is commutative, and then ${}^t x = x^*$).

0.6. A matrix $x = (x_{ij}) \in M_n(R)$ is called alternating if ${}^t x = -x$ and $x_{ii} = 0$. The set of alternating matrices in $M_n(R)$ is denoted by $A_n(R)$.

Assume that R is alternative with unity, and that it has an involution; i.e., an antiautomorphism of period 2 . If R_0 is an ample subspace of R (cf. Jacobson[3]) then we denote by $H_n(R,R_0)$ the set of hermitian matrices in $M_n(R)$ with diagonal entries in R_0 . If $n \leq 3$ or R is associative this is a unital Jordan algebra.

0.7. The notation a.b.c refers to formula (c) in section a.b .