Lecture Notes in Mathematics

A collection of informal reports and seminars
Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

260

Roger Godement
Université de Paris, Paris/France

Hervé Jacquet
The City University of New York,
New York, NY/USA

Zeta Functions of Simple Algebras

Springer-Verlag
Berlin · Heidelberg · New York 1972

AMS Subject Classifications (1970): 10D20, 12A70, 12B35, 22E50

ISBN 3-540-05797-8 Springer-Verlag Berlin · Heidelberg · New York
ISBN 0-387-05797-8 Springer-Verlag New York · Heidelberg · Berlin

Offsetdruck: Julius Beltz, Hemsbach/Bergstr.

Introduction

Two of the best known achievements of Hecke are his theory of L-functions with grossencharacter and his theory of the Dirichlet series associated to automorphic forms on GL(2). Since a grossencharacter is an automorphic form on GL(1) and Hecke's results on L-functions can be best proved by using Tate's technique, it is natural to ask if it is possible to extend Tate's technique to GL(2), or, more generally, to the multiplicative group of an arbitrary simple algebra. This is, of course, not a new question and many authors have given partial answers.

The purpose of this set of notes is to give an affirmative and, in some sense, complete answer. Of course, complete is a relative word. A complete treatment of the question would imply at the very least a complete knowledge of the representations of the local groups. This is not available at the moment. Actually the existence of the "absolutely cuspidal representations" is not even proved in this set of notes.* When a complete list of the irreducible representations of the local groups becomes available, it will presumably be relatively easy to compute the factors attached in this paper to such a representation.

Also in the global theory, I have restricted myself to the case of cusp forms. If my understanding of the theory is correct, the other forms should not give essentially new Euler products. However, this is not to say that it would be without interest to consider other forms as well.

---

*In particular, we do not prove the existence of nontrivial functions in the space of Lemma 5.3.

Since this is a joint work and I alone had responsibility for writing the final version of these notes, deciding on the plan and the contents, and making mistakes, it is perhaps best to explain the genesis of this work. In 1967, R. Godement gave a series of lectures in Princeton and later on, in Tokyo, on the global theory. Chapter II is largely, if not totally, based on the notes he made available to me. At that time, only the local problems remained. In 1969, he gave another series of lectures at the Institute for Advanced Study on "Local zeta functions for square integrable representations". The main substance of those notes is explained in §9. Since we had at our disposal his technique of induction (already exploited in [1] and [14], it was clear that all that was needed, in order to complete the theory, was a theorem asserting that all irreducible representations of linear groups are, in some sense, induced by square integrable ones.

In the p-adic case such a result was available in my Montecatini lectures. Actually one can arrive at a stronger theorem where the notion of square integrable representation is replaced by the notion of"absolutely cuspidal representation" (super cuspidal in the terminology of Harish-Chandra). This is proved in §2 (which is practically extracted from my Montecatini lectures). One is therefore reduced to the case of an absolutely cuspidal representation which is treated in §4 and §5. Actually the case of a division algebra treated in §4 could have been included in §5. But I thought that a separate treatment would throw some light on the theory. The case of a split simple algebra and absolutely cuspidal representation is taken up in §5. The method that I follow here differs from the one explained in §9 (Godement's method).

Since both methods seem to be equally natural, both are included. To illustrate our principles, I have treated completely, not only the unramified case, which is indispensable for the global theory, but also the case of the "special representation". I have found convenient to give a self-contained account of this representation. The forthcoming work of A. Borel and J. P. Serre should make the proof of Theorem 7.11 more elegant if not more simple. In general, the sections of this work devoted to representations of p-adic groups are self-sufficient with the single but notable exception of Section 6.

In the archimedean case the induction technique and a theorem of Harish-Chandra reduce the local theorem to the case of a division algebra which is already known. Since no new idea is involved I give only the briefest account of the theory. The reader will observe that here again this work is still incomplete. One should certainly be able to classify all irreducible representations of the real and complex groups and compute explicitly the factors attached to them. Presumably, it would appear then that the notion of archimedean Euler factor given here is too general and that a more restricted definition should be used.

No doubt that the results developed here will someday disappear in the general theory of Euler products associated to automorphic forms. No doubt also that this work is, at the moment, incomplete. But we feel that its present publication could be of some use to the mathematical community.

It remains only for me to thank the University of Maryland, The Scuola Normale Superiore of Pisa, the Centro Internazionale Matematico

Estivo and the Graduate Center of The City University of New York whose hospitality made this work possible. It is also a pleasure to extend my thanks to Mrs. Sophie Gerber who typed these notes with her usual expertness.

In the Bibliography, I have tried perhaps not successfully to indicate our indebtedness to other authors. The reader will find there a list of technical references arranged in the order in which they are used in the paper as well as a list, probably only partial, of previous papers on the same subject. I wish also to thank G. Shimura for a pertinent remark on Chapter I which spared me the embarrassment of a serious mistake.

Hervé Jacquet

New York, December 1971

Notations

To help the reader we give a partial list of the notations used here. "Bold face" characters are replaced by underlined capitals. Thus $\underline{Z}$, $\underline{R}$, $\underline{C}$, $\underline{H}$ stand for the ring of rational integers, the field of rational numbers, the field of real numbers, the field of complex numbers, the ring of Hamilton quaternions.

In Chapter I and II the ground field is denoted F . It is a local field in Chapter I and a global field in Chapter II. We consider a simple algebra M of center F (actually in §2, for technical reasons we had to consider a semi-simple algebra). Its rank is n^2 . Its reduced norm is denoted ν_M or simply ν and its reduced trace τ_M or simply τ . The multiplicative group of M can be regarded as an algebraic group defined over F . It is denoted by G .

When F is local we denote by mod_F or α_F its topological module, the module of an element being denoted $|x|_F$ or $|x|$ when this does not create confusion. We choose a nontrivial additive character ψ_F or ψ of F . Then dx denotes the self-dual Haar measure on M with respect to $\psi_F \circ \tau$. On the other hand $d^\times x$ denotes any Haar measure on G_F (unless the Haar measure is otherwise specified). Of course, in the functional equation, the same measure is used on both sides. When F is nonarchimedean we denote by R_F or R the ring of integers in F , by q the cardinality of the residual field and by v_F or v the normalized valuation. Hence $|x|_F$ is $q^{-v(x)}$. We consider "admissible representations" of a certain algebra associated to the group G_F (in the nonarchimedean case, they may be regarded as representations of G_F). Some of them are called absolutely cuspidal.

This notion is equivalent to the notion of super cuspidal representations given by Harish-Chandra (although this takes a proof). The contragredient to such a representation π is defined. It is noted $\tilde{\pi}$. To π we associate the Euler factors $L(s,\pi)$ and $L(s,\tilde{\pi})$ as well as the factor $\epsilon(s,\pi,\psi_F)$. We define also

$$\epsilon'(s,\pi,\psi_F) = \epsilon(s,\pi,\psi_F)L(1-s,\tilde{\pi})/L(s,\pi) .$$

For GL(1) and GL(2) the factors L and ϵ coincide with the ones introduced in [1].

In Chapter II the ground field F is global and we follow standard notations. For instance $\underline{A}$ is the ring of adeles and $\underline{I}$ the group of adeles. A place of F is denoted by the symbol v . Then F_v is the corresponding local field and $M_v = M \otimes F_v$, $G_v = M_v^\times$.

Table of Contents