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C. Gruber  
A. Hintermann  
D. Merlini

Group Analysis  
of Classical Lattice Systems

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## Authors

C. Gruber

Lab. Phys. Théorique

Ecole Polytechnique Fédérale

Case Postale 1024

Lausanne/Schweiz

A. Hintermann

S.I.N.

Villigen

D. Merlini

C.R.P.P.

Ecole Polytechnique Fédérale

Case Postale 1024

Lausanne/Schweiz

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## F O R E W O R D

In 1969 a theoretical Physics Group was created at the Physics Department of the Federal Institute of Technology of Lausanne. In those days it was decided to initiate research on selected topics of Statistical Physics, in particular on the statistical mechanics of Lattice Systems. C. Gruber, a Ph. D. from Princeton University, who had joined the Group that same year, conducted this part of the research programme. In 1971 two graduate students from E.T.H. Zurich, A. Hintermann and D. Merlini started to work on their Ph.D. thesis with C. Gruber as thesis advisor.

In 1975, G. Gallavotti and G. Jona-Lasinio, who had been invited to be members of the thesis committee, suggested to present in one monograph the work of Gruber, Hintermann and Merlini. The idea was accepted with enthusiasm and an optimistic planning was laid down. After a finite sequence of lower bound estimates to its duration, the project came ultimately to completion.

I would like to take this opportunity to congratulate the authors who have proved to successfully overcome the difficulties of all kind inherent to such enterprise and I wish for them an encouraging acceptance of their monograph by the audience interested in the subject.

Ph. Choquard

Lausanne, December 20, 1976

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