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Abstract Analytic Function Theory and Hardy Algebras

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Preface

The present work wants to be the systematic presentation of a functional-analytic theory. It is an abstract version of those parts of classical analytic function theory which can be circumscribed by boundary value theory and Hardy spaces H^p. The fascination of the field comes from the fact that famous classical theorems of typical complex-analytic flavor appear as instant outflows of an abstract theory the tools of which are standard real-analytic methods such as elementary functional analysis and measure theory. The abstract theory started about twenty years ago in papers of Arens and Singer, Gleason, Helson and Lowdenslager, Bochner, Bishop, Wermer,... and went through several steps of abstraction (Dirichlet algebras, logmodular algebras,...). It never ceased to radiate back and illuminate the concrete classical theory. We present the ultimate step of abstraction which has been under work for about ten years.

The central concept is the abstract Hardy algebra situation. It is comprehensive as well as pure and simple and permits to build up a coherent theory of remarkable and pleasant width and depth. We attempt to present a systematic account in Chapters IV-IX. The abstract Hardy algebra situation can be looked upon as a local section of the abstract function algebra situation. To achieve the localization is the main business of the abstract F.and M.Riesz theorem and of the resultant Gleason part decomposition procedure. These are central themes in Chapters II-III devoted to the abstract function algebra situation. Chapter I presents the concrete unit disk situation in such a spirit as to prepare the abstract concepts. Chapter X is devoted to standard applications of the abstract theory to polynomial and rational approximation in the complex plane and is the most conventional part of the book.

In comparison with the respective parts of the earlier treatises on uniform algebras, the most comprehensive of which is GAMELIN [1969], the present work contains numerous new results. In concepts and systematization it is shaped after the work of König. Most of the chapters contain substantial new material. A prime point is the systematic use of the associated algebra $H^{\#}$. The most important individual new result is perhaps the approximation theorem VI.4.1. Let us also quote Section VI.5 on the Marcel Riesz estimation for the abstract conjugation after fundamental results of Pichorides in the unit disk situation. For more details we refer to the Introductions and Notes to the individual chapters.

In its overall structure and in certain parts the present work resembles the lectures on function algebras which König held in 1967/68 at the California Institute of Technology in Pasadena/California, and which in part had been distributed in a provisional form. He wants to express his warmest thanks to Wim Luxemburg and Charles DePrima who were his hosts in those days, and likewise to Gunter Lumer to whom he owes the participation in the Function Algebra Seminar at the University of Washington in Seattle/Washington in 1970. Above all he sends his deepest thanks to Galen Seever and to Kôzô Yabuta for most valuable and pleasant cooperation, and he wants to include his former student Klaus Barbey who started to participate with the elaboration of 1970/71 lecture notes which formed the next step in the evolution of the present text.

In conclusion we want to express our sincere thanks to Michael Neumann for his active interest and valuable work in connection with a common seminar, to Ulla Faust and Gisela Schirmbeck who typed the final text with impressive care and thoughtfulness, to Horst Loch who read most of the proofs with distinctive care, and to Karla May and Gerd Rodé for their kind assistance.

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