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Self-Adjoint Operators

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PREFACE

These lectures are primarily concerned with the problem of when the sum of two unbounded self-adjoint operators is self-adjoint. The sum fails to be self-adjoint when there is an ambiguity about the choice of boundary conditions. There is then also an ambiguity in the definition of functions of the sum.

This problem is fundamental to quantum mechanics. Kinetic energy and potential energy are self-adjoint operators, and functions of either of these may be computed explicitly. The sum is the total energy, and the main task of quantum mechanics is to compute functions of it. Thus the question of self-adjointness is the question of whether this task is meaningful. If it is, then among other things the dynamics of the system is determined for all states and all times.

There are obvious series expansions for certain important functions, such as the resolvent. The point is that these series may not converge. In order to demonstrate self-adjointness it is necessary to develop alternative methods for approximating functions of the sum.

In the language of physics, the basic question is whether the force laws determine the motion. The models currently used contain elements which could lead to ambiguity. For example, the charge on an electron is considered to be squeezed down to one point. As a consequence its potential energy is unbounded. What happens when two electrons occupy the same point? Their interaction energy is infinite, and it is not clear how they will move.

Actually, according to quantum mechanics it is improbable that two electrons will be at or even near the same point. But there are similar difficulties in the description of light by quantum field theory, and these are not so easy to resolve. The question of whether this theory gives unambigous predictions to an arbitrarily high degree of accuracy remains open. So it is worth while to examine the mechanism for determining the dynamics in better understood situations.

The justification for these lectures is that there has been progress on the self-adjointness problem since the publication of Kato's book, Perturbation Theory for Linear Operators (1966). This progress has been stimulated largely by developments in quantum field theory. While the results which depend on a series expansion have not been significantly improved, those which exploit positivity are the heart of the recent developments. This is perhaps because for many physical systems one expects the total energy to be bounded below.

While Kato's book is the basic reference for linear perturbation theory, there are other books with additional material on the quantum mechanical applications. Those by Hellwig (1967) and Glazman (1965) approach the subject with partial differential equation techniques. More recently new progress has been made on the determination of the spectra of quantum mechanical operators, and this is described in the books of Simon (1971a), Schechter (1971), and Jörgens and Weidmann (1973).

Recently a memoir of Chernoff (1974) has appeared. It contains a valuable discussion of the addition problem which complements the present treatment.

These lectures begin with a review of standard material on addition of self-adjoint operators. The second part is a selection from the more recent developments. This includes a section on properties of eigenvalues, including uniqueness of the ground state. The third part is devoted to the classification of extensions of a Hermitian operator. This is standard material; however it is there to illustrate what kind of ambiguity is possible when self-adjointness fails. (After all, to understand the force of a theorem you have to be able to imagine a situation where its conclusion fails.) The final part is a brief account of how self-adjointness is related to the determination of a measure by its moments.

The core of the lectures is the first two parts. The theme is the interplay between the two aspects of a quantum mechanical observable, as an operator and as a quadratic form. In order to be able to take functions of an observable it is necessary for it to be a self-adjoint operator. But to add observables it is most natural to add quadratic forms (that is, to add expectation values).

Throughout, the theory is illustrated by the case of the Schrödinger equation for a non-relativistic particle in a given potential. The emphasis is on results obtained by operator theory rather than by partial differential equation methods. (There is almost no discussion of the case of ordinary differential equations, but this subject has recently been surveyed by Devinatz (1973).) There is a brief description of the applications to quantum field theory.

These lectures are intended to be an introduction to one topic in operator theory. They are not a complete treatment even of this topic, but should be regarded as an invitation to the research literature. In order to follow them it should be sufficient to know real analysis and have some acquaintance with Hilbert space. The spectral theorem is stated but not proved.

The lectures were first given (in a somewhat different form) at the Eidgenössische Technische Hochschule in Zürich during the spring of 1971. I wish to thank Professor Barry Simon for references to the literature, and Dr. Jean-Pierre Eckmann, Dr. Charles Stuart, and Dr. Lawrence Thomas for reading the manuscript. I am especially grateful to Miss Edeltraud Russo for her excellent secretarial work.

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