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237

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Rings and Modules of Quotients

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Contents

Introduction

These notes are intended to give a survey of the basic, more or less well-known, results in the theory of rings of quotients. An effort has been made to make the account as self-contained and elementary as possible. Thus we assume from the reader only a knowledge of the elements of the theory of rings and of abelian categories.

We will briefly describe the contents of the notes. Chapter 1 treats the necessary preliminaries on torsion theory. The main result here is the establishing of a 1-1 correspondence between hereditary torsion theories and topologies on a ring (Gabriel [31] and Maranda [51]).

In Chapter 2 we construct rings and modules of quotients with respect to an additive topology, following the approach of Gabriel ([10], [31]). This construction is a special instance of the construction of associated sheaves of presheaves for an additive Grothendieck topology; in these notes, however, we will not pursue that course. Rings and modules of quotients are then described in terms of injective envelopes (Johnson and Wong [114] and Lambek [46]). The main result of this chapter is the theorem of Popescu and Gabriel [62] which asserts that every Grothendieck category $\underline{C}$ is the category of modules of quotients for a suitable topology on the endomorphism ring of a generator for $\underline{C}$. The proof we give for this theorem is a simplified version of Popescu's proof [12], due to J. Lambek [48] and J.E. Roos [unpublished].

In Chapter 3 we treat some aspects of rings of quotients related to finiteness conditions. In particular we prove a theorem of Popescu and Spircu [104] which characterizes flat epimorphisms in the category of rings as a special class of rings of quotients. In this context we also discuss rings of fractions, i.e. rings obtained by inverting elements of some multiplicatively closed subset of a ring.

Chapter 4 contains some material on self-injective rings. Various well-known characterizations of quasi-Frobenius rings are given here. These results are used in Chapter 5 where we discuss maximal rings of quotients and classical rings of quotients. Necessary and sufficient conditions for these rings to be regular, semi-simple or quasi-Frobenius are given (results due to Gabriel [31], Goldie [91], Mewborn and Winton [54], Sandomierski [68], and others).

The references at the end of each section are intended to tell the reader where he can find a further discussion of the treated topics. Their purpose is not to record credit for the results of the section.

I am grateful to J.E. Roos for allowing me to use his preliminary manuscripts for [66].

Stockholm, December 1970

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Some notation

All rings have identity elements. A denotes a ring and all modules are right A-modules, unless otherwise is stated. The category of right A-modules is denoted by Mod-A, and we write M_A to indicate that M is in this category.

E(M) is the injective envelope of M_A.

If L is a submodule of M and $x \in M$, then $(L:x) = \{a \in A \mid xa \in L\}$.

J(A), or simply J, denotes the Jacobson radical of A.