Lecture Notes in Mathematics

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PREFACE

L^1-algebras of locally compact groups have been studied for many years. The systematic study of Segal algebras - which are a generalization of L^1-algebras - was begun only a few years ago, for locally compact abelian groups. It is the purpose of these notes to study Segal algebras for general locally compact groups and to discuss also some new results for L^1-algebras. There are many problems here for further research, and it is hoped that the reader will be able to continue where the author had to stop.

The lectures of which these notes are the outcome were delivered at the Universities of Heidelberg, Nancy and Utrecht in 1969 and 1970. I wish to express here my cordial thanks to Professors H. Leptin (Heidelberg) and P. Eymard (Nancy) for their invitations and, especially, to my friends and colleagues in Utrecht who have made my long stay in the Netherlands so pleasant.

Thanks are also due to Professor B. Eckmann of the Eidgenössische Technische Hochschule, Zürich, and to Springer-Verlag for making publication of these lectures possible.

Dr. W. Beiglböck (Heidelberg), Mr. J.-P. Pier (Nancy) and Mr. M. Riemersma (Utrecht) have kindly provided me with their notes of the original lectures; Mr. Riemersma has also examined the manuscript itself and has contributed much helpful criticism. Last but not least, Miss W. Jenner (Utrecht) has typed the whole manuscript with the utmost care. My thanks go to each and all of them.

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READER'S GUIDE

The reader is assumed to be acquainted with the author's monograph 'Classical harmonic analysis and locally compact groups' (Oxford University Press, 1968); terminology, notation and results explained there are used freely here.

References such as Ch. 6, § 2.2, without further specification, refer to the monograph mentioned. References to the Bibliography at the end of the present lecture notes are given by a name in capital letters followed by a number in brackets, e.g. WEIL [1].

The subject-matter of the lectures may be divided into three parts (which are closely interrelated):

(i) A study of certain ideals of $L^1(G)$ associated with closed subgroups of G. Here the property P_1 is shown to play an essential role. This part represents a generalization of results given in REITER [2].

(ii) The investigation of Segal algebras $S^1(G)$ for general locally compact groups. This extends the study of the abelian case in Ch. 6, § 2. Ideals associated with subgroups of G are studied for Segal algebras.

(iii) A general approach to approximate units, especially in closed ideals of $L^1(G)$ or $S^1(G)$. This is connected with recent work in harmonic analysis described in Ch. 7, § 4, and leads to new developments.

A summary of contents follows.

In § 1 we associate with a closed subgroup H of a locally compact group G a closed left ideal and a closed right ideal of $L^1(G)$. These ideals coincide if and only if H is normal; they play a central role in later developments.

§ 2 contains some preliminary applications of the property P_1 to the ideals introduced in § 1.

In § 3 we discuss a modification of the property P_1, and also its relativization (in the sense of Ch. 4, § 5.1). This is needed for later applications.

In § 4 Segal algebras are defined for general locally compact groups; they may be said to constitute a natural extension of L^1-algebras. There are two main classes: symmetric and pseudosymmetric Segal algebras; the former are closer to the abelian case. Some basic properties of Segal algebras are discussed; here integration of functions with values in a Banach space plays a role.

In § 5 examples of Segal algebras are given; some of these present non-trivial problems (cf. examples (iii) and (viii) in § 5).

In § 6 approximate left (right, two-sided) units are introduced, for Banach algebras. This concept, as defined here, is more general than that traditionally used; the significance of the general definition will emerge from the results to be proved in these lectures.

§ 7 contains some lemmas on approximate units of various kinds, in Banach algebras. These lemmas are required for later applications; the proofs are somewhat technical and may well be omitted on a first reading.

In § 8 approximate units in Segal algebras are considered. It is shown that some familiar properties of $L^1(G)$ carry over to Segal algebras, but the proofs are more subtle than in the classical case of $L^1(G)$.

§ 9 is devoted to a general study of closed ideals in Segal algebras. It is proved that the bijective correspondence between the closed ideals of a Segal algebra $S^1(G)$ and those of $L^1(G)$, established in Ch. 6, § 2.4, for locally compact abelian groups, extends to all symmetric Segal algebras and in part even to pseudosymmetric ones - a result due, essentially, to BURNHAM [1,2], in an even more general formulation.

The foundations having been laid, the results that form the main aim of these lectures can now be established.

In § 10 closed ideals associated with closed subgroups H of G are introduced for symmetric, and in part also for pseudosymmetric, Segal algebras $S^1(G)$, in analogy with the definition given for $L^1(G)$ in § 1. Two theorems are proved which establish a close connection between the existence of approximate units in these ideals and the property P_1 of the subgroup H.

In § 11 the associated ideals are further studied, by two methods; one of these is the method of linear functionals which plays a considerable role later (§ 16).

In § 12 normal subgroups and their associated ideals in symmetric Segal algebras are considered. Here the theorems of § 10, combined with the results of § 11, show their full power, yielding the following corollary, among others: the associated ideal of a closed, normal subgroup H has approximate left (or right) units, bounded in the L^1-norm, if and only if H has the property P_1; moreover, if this is the case, then for a large class of Segal algebras the associated ideal has approximate units with special properties which are of interest even in the abelian case.

In § 13 it is shown that a Segal algebra $S^1(G)$ gives rise, in a natural way, to Segal algebras $S^1(G/H)$ for quotient groups of G, and that various properties of $S^1(G)$ carry over to $S^1(G/H)$. It was proved in Ch. 8, § 4.6 (ii), that the image of a closed ideal of $L^1(G)$ under the natural morphism $L^1(G) \to L^1(G/H)$ is a closed ideal of $L^1(G/H)$, if H has the property P_1; this result is extended here to symmetric and pseudosymmetric Segal algebras.

§ 14 is devoted to the actual construction of closed ideals possessing various kinds of approximate units, in symmetric and pseudosymmetric Segal algebras. Here abelian groups and soluble groups are considered; the general lemmas on approximate units in § 7 play a basic role in this context, jointly with the results in §§ 12 and 13.

In § 15 it is proved that for symmetric Segal algebras on compact groups all (two-sided) ideals have approximate units. Here, of course, some special features appear, and also some analogies with the abelian case. As an application, Segal algebras on semidirect products of compact and locally compact abelian groups are considered and the construction of ideals with various kinds of approximate units in these Segal algebras is discussed, in analogy with the method of § 14.

In § 16 Segal algebras $S^1(G)$ on abelian groups G are studied. It is proved that in this case the bijective correspondence between the closed ideals of $S^1(G)$ and those of $L^1(G)$ established in Ch. 6, § 2.4 (and discussed again in § 9 of these lectures) has the following property: to closed ideals with approximate units in $S^1(G)$ correspond closed ideals with approximate units in $L^1(G)$, and vice-versa. In the proof - which is based on the method of linear functionals, already used in § 11 - the abelian groups structure enters decisively at one particular point; it is an open question whether the result holds for groups that are not abelian (nor compact, the compact case having been settled in § 15). In connection with this result, Wiener-Ditkin sets in abelian groups (Ch. 7, § 4) are discussed, in particular the injection theorem, the proof of which is given here in an improved form.

§ 17 is concerned with the investigation of those closed (two-sided) ideals in $L^1(G)$, for compact groups G and for locally compact abelian groups, which have approximate units of a particular kind corresponding essentially to approximate units in the traditional sense. Here the situation is particularly satisfactory: the structure of such ideals can be determined and the analogy between compact and locally compact abelian groups appears in a lucid way. Finally, some open questions are mentioned which present themselves quite naturally in this context.

These and other questions raised in the course of these lectures, and the further study of Segal algebras, must be left to the reader's own research.

§ 18 contains additions and corrections to the author's monograph mentioned at the beginning.

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