Lecture Notes in Mathematics

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Etale Homotopy

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These notes are an expansion of the results announced in [2]. The material was presented in a seminar at Harvard University during the academic year '65-'66.*

Our aim is to study the analogues of homotopy invariants which can be obtained from varieties by using the etale topology of Grothendieck. Using the constructions of Lubkin [21] or Verdier [3], we associate to any locally noetherian prescheme X a pro-object in the homotopy category of simplicial sets (cf. 2), which we call the etale homotopy type of the prescheme X. For a normal variety over the field of complex numbers, we show that X_{et} is a certain profinite completion of the classical homotopy type. This comparison result, together with a number of others, is in section 12.

Much of our work consists in setting up a reasonable theory of homotopy for pro-simplicial sets. This homotopy category turns out to be quite amenable to the techniques of classical algebraic topology. One may establish the analogues of Hurewicz and Whitehead theorems (4.3), (4.4), (4.5), and also one has available the techniques of Postnikoff decomposition. There is

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*We thank G. Borkowski for her flawless typing of the manuscript.

a direction in which this pro-category is more flexible than the classical homotopy category: It is sufficiently rich so as to admit pro-finite and p-adic completion functors (cf. (3.4)). Many of our results about schemes are phrased in terms of these functors.

For instance, let X be a connected, pointed scheme over an algebraically closed field K of characteristic zero, and let X^1, X^2 be the schemes over the complex numbers obtained from X via two embeddings of K in $\mathbb{C}$. Then

$$\hat{X}^1_{cl} \approx \hat{X}^2_{cl} ,$$

where the subscript cl denotes the classical homotopy type, and where $\wedge$ denotes pro-finite completion. We remark that there is an example due to Serre [28] of a projective non-singular variety X over a number field K and imbeddings of K into $\mathbb{C}$ such that X^1_{cl} and X^2_{cl} have non-isomorphic fundamental groups. Thus X^i_{cl} do not necessarily have the same homotopy type. However, as Serre indicated to us, the homotopy type of a nonsingular simply connected projective surface is independent of complex imbedding. This gives an affirmative response in that case to a question raised in [2]. We do not know whether the same holds for higher dimensional simply connected nonsingular varieties.

Atiyah has suggested analogous questions for bundles: Let L/K be an extension of algebraic number fields, with K a subfield of the complex numbers. Let V denote a variety defined over K, and E an algebraic vector bundle defined over $V \otimes_K L$. An imbedding c of L/K into $\mathbb{C}$ over K will induce a complex vector bundle E^c over the topological space $V_{cl} = (V \otimes_K \mathbb{C})_{cl}$. The problem is to study how the class of E^c in $K(V_{cl})$ depends on c.

As another example, let X,Y be smooth proper connected schemes over a field K of characteristic zero. Suppose given <u>two</u> discrete valuation rings R_p, R_q of K with residue fields of characteristics $p \neq q$ respectively, and that X,Y have isomorphic non-degenerate reductions modulo each of the valuations. Let X^1, Y^1 be the schemes over the complex numbers obtained from some imbeddings of K in C, and assume finally that X^1, Y^1 are simply connected. Then

$$\hat{X}^1_{cl} \approx \hat{Y}^1_{cl} .$$

Note that in this case we may conclude that the classical homotopy groups of X^1 and of Y^1 are isomorphic (abstractly) since they are finitely generated abelian groups with isomorphic profinite completions

This result is in the spirit of the question originally posed by Washnitzer: Given two varieties in characteristic zero with isomorphic reductions modulo p, what additional conditions will imply that the varieties are themselves homotopic, or, to go even further, diffeomorphic? We are led to the abstract question: How much information is lost when one passes from an honest simplicial set K to its profinite completion

K ? Suppose that K is of the homotopy type of a finite CW-complex. Then we conjecture that there is at most a finite number of homotopically distinct simplicial sets whose profinite completion is isomorphic to K . In section 7, we prove a stable version of the conjecture, and moreover, we give a complete algebraic description of the set of distinct stable homotopy types with isomorphic profinite completions. We show by examples that there exist distinct stable homotopy types whose completions are isomorphic.

If V is a variety over a field K , and $G = \mathrm{Gal}(\overline{K}/K)$ where $\overline{K}$ is the separable algebraic closure, then G operates on the ℓ-adic homotopy type of $\overline{V} = V \otimes_K \overline{K}$. If V is an abelian variety, this representation yields no further information than the operation on ℓ-adic cohomology. However, for a more general V , it may be expected that more precise information is contained in this representation. Perhaps the special case $V = \mathbb{G}^{m,n}$, the Grassmanian of m-dimensional planes in n-space, $K = \mathbb{Q}$, is of interest.

Indeed, if $\wedge$ denotes ℓ-adic completion, one obtains using the comparison theorem (12.9) an operation of $G = \mathrm{Gal}(\overline{Q}/Q)$ on the pro-simplicial set $\widehat{\mathbb{G}_{\mathbb{C}}^{m,n}}$. In other words, the topological space $\mathbb{G}_{\mathbb{C}}^{m,n}$ is endowed with a natural action of G on its ℓ-adic completion. The recent ideas of Quillen [13](b) with regard to the conjecture of Adams suggest that an analysis of the above action may provide fruitful results in topological K-theory.

We have not attempted to develop an analogue of topological

K-theory associated to the etale topology. Quillen's ideas cry out for the development of such a functor, however. Looking even further ahead, one may wonder whether analogues of the Novikov-Browder invariants (or the more recent theory of Sullivan) describing the diffeomorphy types representing a given tangential homotopy type are amenable to definition for an arbitrary scheme, proper and smooth over a field. This would bring one closer towards an understanding of the above problem of Washnitzer.

We should like to signal two errors in our announcement [2]. Namely, (5.4) of [2] is obvious nonsense. A correct statement is (8.18) below. Theorem (6.3) of [2] lacks a cohomological dimension hypothesis, as in (12.5).