

# Lecture Notes in Mathematics

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Janos Galambos  
Samuel Kotz

## Characterizations of Probability Distributions

A Unified Approach with an Emphasis  
on Exponential and Related Models

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## PREFACE

The stimulus for this research project was supplied during the 1974 Calgary Conference on *Characterizations of Distributions* organized by G.P. Patil; both of us were active participants in this conference and could not help but notice the multitude of results on characterizations widely scattered in the literature.

Our aim in writing this monograph is twofold. One is to bring together the results in this useful and rapidly growing field in order to encourage further fruitful research in this area. We also have in mind a somewhat different objective, which in our opinion, is not less important and urgent at this time of rapid and unprecedented growth in periodical literature. The field of characterizations was independently developed in different branches of applied probability and pure mathematics which inevitably resulted in a certain amount of duplication of efforts. Moreover the terminology used in different sub-disciplines by various authors is not the same. Our second aim is therefore to unify the existing theory. We found that seemingly unrelated mathematical theorems turned out to be part of the same general theory. We therefore tried to discover the unifying thread which passes through the majority of the papers written in the past one and a half decades in the field of characterizations of statistical distributions (in particular those dealing with the exponential distribution, both univariate and multivariate, and its monotonic transformations, which is the subject of this monograph). However, we should point out that there are still a number of as yet unsolved problems, for example, those stemming from Rossberg's results which require additional investigation.

This monograph contains no overlap with existing books on characterizations, since earlier books concentrated mainly on normality, which we do not discuss in our book. Moreover, as a rule, results dealing with exponential distributions which have appeared previously in other monographs or books are not reproduced here. This is true in particular for the material on this subject contained in Johnson and Kotz' (1972) *Multivariate Continuous Distributions* and the recent book by Galambos (1978) *The Asymptotic Theory of Extreme Order Statistics*, both of which were published by John Wiley and Sons.

To make the work self-contained, we have included the derivation of a number of results dealing with basic distribution theory, in particular those related to the distribution of order statistics. On the other hand, not all the theorems stated are supplemented with detailed proofs. In many instances, it was sufficient to present a comprehensive proof of the basic "unifying" theorem from which the subsequent results follow almost directly or require only minor modifications of the argument.

The level of mathematics was kept to a minimum, but the rigor of mathematical analysis was scrupulously maintained. Moreover, we have included extensive comments on the "practicality" of various mathematically equivalent characterizations. We thus feel that the monograph, which contains an up-to-date bibliography of over 250 items, will be useful and of interest to the general mathematical community, to probabilists and statisticians as well as to researchers and practitioners of industrial engineering and operations research and various scientists specializing in the natural and behavioral sciences, in particular those who are interested in the foundations and applications of probabilistic model building.

Philadelphia, Pa.  
April, 1978.

Janos Galambos; Samuel Kotz

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The authors are also indebted to their colleagues throughout the world - too numerous to mention individually - who generously supplied us with reprints and on occasion preprints of their work on characterizations of distributions and related topics which facilitated our task of compiling this monograph.

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