Lecture Notes in Mathematics

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Bessel Polynomials

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To

ELIZABETH

BLANCHE

and

VIVIAN

FOREWORD

The present book consists of an Introduction, 15 Chapters, an Appendix, two Bibliographies and two Indexes.The chapters are numbered consecutively, from 1 to 15 and are grouped into four parts, as follows:

Part I - A short historic sketch (1 Chapter) followed by the basic theory (3 Chapters);

Part II - Analytic properties (5 Chapters);

Part III - Algebraic properties (4 Chapters);

Part IV - Applications and miscellanea (2 Chapters).

According to its subject matter, the chapter on asymptotic properties would fit better into Part II; however, some of the proofs require results obtained only in Chapter 10 (properties of zeros) and, for that reason, the chapter has been incorporated into Part III.

The Appendix contains a list of some 12 open problems.

In the first bibliography are listed all papers, monographs, etc., that could be located and that discuss Bessel Polynomials. It is quite likely that, despite all efforts made, absolute completeness has not been achieved. The present writer takes this opportunity to apologize to all authors, whose work has been overlooked.

A second, separate bibliography lists books and papers quoted in the text, but not directly related to Bessel Polynomials.

References to the bibliographies are enclosed in square brackets. Those referring to the second bibliography are distinguished by heavy print. So [1] refers to: W.H. Abdi - A basic analog of the Bessel Polynomials; while [**1**] refers to: M. Abramowitz and I.E. Segun - Handbook of Mathematical Functions.

Within each chapter, the sections, theorems, lemmata, corollaries, drawings, and formulae are numbered consecutively. If quoted, or referred to within the same chapter, only their own number is mentioned. If, e.g., in Chapter 10 a reference is made to formula (12), or to Section 2, this means formula (12), or Section 2 of Chapter 10. The same formula, or section quoted in another chapter, would be referred to as formula (10.12), or Section (10.2), respectively. The same holds, mutatis mutandis, for theorems, drawings, etc.

While writing this book, the author has received invaluable help from many colleagues; to all of them he owes a great debt of gratitude. Of particular importance was the great moral support received from Professors H.L. Krall and O. Frink, as well as A.M. Krall. Professors Krall also read most of the manuscript and made valuable suggestions for improvements.

As already mentioned, there is no hope for an absolutely complete bibliography; however, many more omissions would have occurred, were it not for the help received, in addition to the mentioned colleagues, also from Professors R.P. Agarwal, W.A. Al-Salam, H.W. Gould, M.E.H. Ismail, C. Underhill, and A. Wragg.

Last, but not least, thanks are due to Ms. Gerry Sizemore-Ballard, for her skill and infinite patience in typing the manuscript and to my daughter Vivian for her help with the Indexes.

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July 1978 E. Grosswald

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INTRODUCTION

Let us look at a few problems that, at first view, have little in common.

PROBLEM 1: To prove that if $r = a/b$ is rational, then e^r is irrational; also that π is irrational.

Following C.L. Siegel [**53**] (who streamlined an idea due to Hermite), one first determines two polynomials $A_n(x)$ and $B_n(x)$, both of degree n, such that $e^x+A_n(x)/B_n(x)$ has a zero of order (at least) $2n + 1$ at $x = 0$. This means, in particular, that the power series expansion of $R_n(x) = B_n(x)e^x + A_n(x)$ starts with the term of degree $2n + 1$, $R_n(x) = c_1x^{2n+1} + c_2x^{2n+2} + \dots$ say. By counting the number of conditions and the number of available coefficients, it turns out that $A_n(x)$ and $B_n(x)$ are uniquely defined, up to a multiplicative constant. By proper choice of this constant one can obtain that $A_n(x)$ and $B_n(x)$ should have integer coefficients. By simple manipulations one shows that $A_n(-x) = -B_n(x)$ and that

$R_n(x) = (n!)^{-1}x^{2n+1} \int_0^1 t^n(1-t)^n e^{tx}dt$. The last assertions are proved by effective construction of the polynomials involved (see Sections 14.2 and 14.3 for details). It follows from the integral representation that $|R_n(x)| \le (n!)^{-1}|x|^{2n+1}e^{|x|}$ and that $R_n(x) > 0$ for $x \ge 0$. If now $e^r = e^{a/b}$ were rational, also e^a would be rational; let $q > 0$ be its denominator. As already observed, $B_n(a)$ and $A_n(a)$ are integers, so that

$$m = qR_n(a) = q(B_n(a)e^a + A_n(a)) \text{ is a positive integer.}$$

Using the bound on $R_n(a)$, $0 < m < q.(n!)^{-1}a^{2n+1}e^a$ and, by Stirling's formula, $0 < m < q(a^{2n+1}e^a/n^{n+1/2}e^{-n}(2\pi)^{1/2})(1+\varepsilon)$, where $\varepsilon \to 0$ as $n \to \infty$. For sufficiently large n, $0 < m < 1$, which is absurd, because m is an integer. Hence, e^r cannot be rational.

Next setting $x = \pi i$, $R_n(\pi i) = -A_n(-\pi i)(-1) + A_n(\pi i)$

$$= (-1)^{n+1} \frac{\pi^{2n+1}}{n!} \int_0^1 t^n(1-t)^n \sin \pi t \, dt$$

(the last equality depends on some computations and will be justified in Chapter 14). The integrand is positive, so that $R_n(\pi i) \neq 0$. Let $k = [\frac{n}{2}]$ where $[x]$ stands for the greatest integer function; then $A_n(x) + A_n(-x)$ is a polynomial in x^2 of degree k and with integer coefficients. Hence, if π^2 is rational, with denominator $q > 0$, then $q^kR_n(\pi i) = q^k\{A_n(\pi i) + A_n(-\pi i)\} = m$, an integer, possibly negative, but certainly

$\neq 0$. Also, by using the integral representation of $R_n(x)$, $0 < |m| = q^k |R_n(\pi i)| \leq (n!)^{-1} q^k \pi^{2n+1} < \frac{(q^{1/2}\pi^2)^n \pi}{n^{n+1/2} e^{-n} (2\pi)^{1/2}} (1+\varepsilon)$ ($\varepsilon \to 0$ for $n \to \infty$), or $0 < |m| < 1$ for sufficiently large n. This is, of course impossible for integral m. Hence π^2, and a fortiori π are irrational. A (highly nontrivial) modification of this proof permits one to show much more, namely that e^r is actually transcendental for real, rational r. In particular, for r = 1, this implies the transcendency of e itself.

PROBLEM 2. To prove that the Student t-distribution of 2n+1 degrees of freedom is infinitely divisible.

We do not have to enter here into the probabilistic relevance, or even into the exact meaning of this important problem. Suffice it to say that, based on the pioneering work of Paul Levy [44] and of Gnedenko and Kolmogorov [19], Kelker [65] and then Ismail and Kelker [60] proved that the property holds if, and only if the function $\phi(x) = \frac{K_{n-1/2}(\sqrt{x})}{\sqrt{x}\, K_{n+1/2}(\sqrt{x})}$ is completely monotonic on $[0,\infty)$, which means that $(-1)^k \phi^{(k)}(x) \geq 0$ for $0 < x < \infty$ and all integral $k \geq 0$. Now, it is well-known (see, e.g. [1], 10.2.17) that if the index of $K_\nu(z)$ (the so called modified Hankel function) is of the form n+1/2 (n an integer), then $(2z/\pi)^{1/2} e^z K_{n+1/2}(z) = p_n(1/z)$, with $p_n(u)$ a polynomial of exact degree n. Previous relation can now be written as $\phi(x) = \frac{p_{n-1}(x^{-1/2})}{x^{1/2} p_n(x^{-1/2})}$, or, with $P_n(u) = u^n p_n(1/u)$, $\phi(x) = \frac{P_{n-1}(x^{1/2})}{P_n(x^{1/2})}$. We now use Bernstein's theorem (see [68]); this asserts that $\phi(x)$ is completely monotonic if, and only if it is the Laplace transform of a function G(t), non-negative on $0 < t < \infty$. In the present case it is possible to study the polynomials $P_n(x)$ and compute G(t). It turns out that $G(t) \geq 0$ for small $t \geq 0$ and also for t sufficiently large. This alone is not quite sufficient to settle the problem, but if we also knew that G(t) is monotonic, then the conclusion immediately follows. In fact, by playing around with G(t), one soon suspects that it is not only monotonic, but actually completely monotonic. In order to prove this, one appeals once more to Bernstein's theorem and finds that G(t) is the Laplace transform of $\phi(x) = (\pi^2 x)^{-1/2}\{1 + \sum_{j=1}^{n} \alpha_j (x + \alpha_j^2)^{-1}\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the zeros of the polynomial $P_n(u)$. A detailed study of these zeros permits one to reduce the large bracket to the form $x^n/q(x)$, where q(x) is a polynomial with real coefficients and such that $q(x) > 0$ at least for $x > -\min_{1 \leq j \leq n} |\alpha_j|^2$. This shows that $\phi(x) > 0$ for

$0 < x < \infty$; hence, $G(t) \geq 0$ on $0 < t < \infty$ and $\phi(x)$ is indeed completely monotonic, as we wanted to show.

PROBLEM 3. To solve the equation with partial derivatives

(1) $\Delta V = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$ where Δ is the Laplacian,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \cot\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2},$$

with the boundary conditions (i) and (ii):

(i) $V = V(r,\theta,\phi,t)$ is symmetric with respect to a "polar axis" through the origin, so that, in fact, $V = V(r,\theta,t)$ only;

(ii) V is monochromatic, i.e., all "waves" have the same frequency ω; and with the initial condition

(iii) the values of V are prescribed along the polar axis at $t = 0$, say $V(r,0,0) = f(r)$, a given function of r.

Here r, θ, and ϕ are the customary spherical coordinates, t stands for the time and c represents dimensionwise a velocity.

Conditions (i) and (ii) are imposed only in order to simplify the problem and can be omitted, but the added complexity can easily be handled by classical methods and has nothing to do with the problem on hand.

Following the lead of Krall and Frink [68] we look, in particular, at solutions of (1) of the form (obtained by separation of variables)

$$u = r^{-1}y(1/ikr)L(\cos\theta)e^{ik(ct-r)}.$$

Here y and L are, so far, undetermined functions and we shall determine them precisely by the condition that u be a solution of (1), while k is a parameter related to the frequency ω by $k = \omega/c$. On account of (ii), k is a well defined constant. The (artificially looking) device of introducing complex elements into this physical problem is useful for obtaining propagating, rather then stationary waves. The real components v, w of $u = v + iw$ will be real solutions of (1) and represent waves traveling with the velocity c. For $c = 0$ and with $x = 1/ikr$, one obtains, of course, directly a real, stationary solution of (1). We now substitute u into (1), by taking into account that $\partial u/\partial\phi = 0$, and obtain, with $x = 1/ikr$ and $z = \cos\theta$, that

$$L(z)(x^2y''(x) + (2+2x)y'(x)) + y(x)((1-z^2)L''(z) - 2z\,L'(z)) = 0,$$

or, equivalently,

$$\frac{x^2y''(x)+(2+2x)y'(x)}{y(x)} = -\frac{(1-z^2)L''(z)-2zL'(z)}{L(z)}.$$

These two functions, each of which depends on a different independent variable, can be identically equal only if their common value reduces to a constant, say C. It follows that L(z) satisfies an equation of the form

$$(1-z^2)L''(z)-2z\,L'(z) + CL(z) = 0.$$

We immediately recognize here the classical equation of Legendre. If, but only if $C = n(n+1)$ with n an integer (it is clearly sufficient to consider only $n \geq 0$, because $-n(-n+1) = n(n-1)$) does this differential equation admit a polynomial solution, namely the Legendre polynomial of exact degree n; we shall denote it by $L_n(z)$. Incidentally, if we would not require symmetry with respect to the polar axis, then we would obtain here the associate Legendre polynomials $P_n^{(q)}(z)$ instead of the simpler Legendre polynomials, and this is the main reason for the present, more restrictive formulation of the problem.

So far, everything has been fairly routine; now, however, it turns out rather surprizingly that, with $C = n(n+1)$ the equation

$$x^2\,y''(x) + (2+2x)\,y'(x) - n(n+1)\,y(x) = 0, \tag{2}$$

satisfied by y(x), also admits a polynomial solution for n an integer, namely a polynomial of exact degree n, uniquely determined up to an arbitrary multiplicative constant. We shall denote it by $y_n(x)$ and may normalize it, e.g., by setting $y_n(0) = 1$.

We have, herewith, obtained a sequence of solutions to (1), of the form

$$u_n = u_n(r,\theta,t) = r^{-1}L_n(\cos\theta)y_n(1/irk)e^{ik(ct-r)}.$$

With each solution u_n and for each constant a_n, also a_nu_n is a solution of (1), and so is the sum $V = \sum_{n=0}^{\infty} a_nu_n$, if it converges. In particular, along the polar axis, with $z = \cos\theta = 1$, $L_n(1)$ is equal to 1, and we obtain at t = 0, with previous substitution $x = 1/ikr$,

$$V = V(r,0,0) = \sum_{n=0}^{\infty} a_n r^{-1}e^{-ikr}y_n(1/ikr) = ik\sum_{n=0}^{\infty} a_n xe^{-1/x}y_n(x).$$

In order to satisfy also the initial condition, we define F(x) by $f(r) = f(1/ikx) = ikx\,F(x)$, so that condition (iii) becomes

$$\sum_{n=0}^{\infty} a_n e^{-1/x}y_n(x) = F(x).$$

From (2) it follows that, by taking as closed path of integration the unit circle,

$$\oint y_n(z)y_m(z)e^{-2/z}dz = \delta_{mn}\frac{2(-1)^{n+1}}{2n+1},$$

where the Kronecker delta $\delta_{mn} = 1$ for $m = n$, $\delta_{mn} = 0$ otherwise. It follows that

$$a_n = (-1)^{n+1}(n+1/2)\oint F(z)y_n(z)e^{-1/z}dz.$$

With these values for a_n, $V(r,\theta,t) = \sum_{n=0}^{\infty} a_n r^{-1}L_n(\cos\theta)y_n(1/ikr)e^{ik(ct-r)}$ is a formal solution of (1), in general complex valued, that satisfies formally all boundary and initial conditions of the problem. It is an actual solution, if the infinite series converges. Precise conditions (that depend on the nature - especially the singularities - of F(z)) are known (see [13]) for this convergence and will be discussed in Chapter 9. Here we add only that in the more general situation, when we discard the restrictions (i) and (ii), the corresponding solution is of the form

$$V(r,\theta,\phi,t) = \sum_k \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{n,k}P_n^m(\cos\theta)\ \sin(m\phi+\phi_0)e^{ik(ct-r)}y_n(1/ikr)/r,$$

with the outer sum extended over all values of $k = \omega/c$, corresponding to all frequencies ω that occur.

What do these problems have in common? All three depend on the study of certain sequences of polynomials, $A_n(x)$ $(= -B_n(-x))$ in Problem 1, $P_n(x)$ in Problem 2, $y_n(x)$ in Problem 3. In fact, the three problems have more in common than just that, because actually, all three sequences of polynomials are essentially the same sequence. There are still many other problems, in which this particular sequence of polynomials known to-day as Bessel Polynomials plays a fundamental role. It also turns out that these polynomials exhibit certain symmetries that are esthetically appealing and have therefore been studied for their own sake. To-day there exists a fairly extensive literature devoted to this specific subject. Nevertheless, recently, when the present author needed some information concerning these polynomials, it turned out that it required an inordinate amount of time to search through a large number of papers and several books, in order to locate many a particular fact needed. It is the purpose of the present monograph, to give a coherent account of this interesting sequence of polynomials. It may be overly optimistic to hope that everything known about them will be found here, but at least the more important theorems will be stated and proved. Originally an attempt was made to obtain all important properties in a unified way, but this attempt has not always been successful; in fact it could hardly have been expected to be. After all, it is not surprizing that the structure of the Galois group of $P_n(x)$ requires for its determination other methods than, say, the study of the domain of convergence of an expansion in a series of these same polynomials.

The author has made himself a modest contribution to the subject matter, but the aim of this work is primarily expository: to systematize and to make easily accessible the work of all mathematicians active in this field. But mainly, unless this book succeeds in relieving the future student of this subject of the need for an exasperating, time consuming search for known items, deeply hidden in the litera ture, it will have failed in its purpose.