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Daniel G. Quillen

Massachusetts Institute of Technology
Cambridge, Mass.

Homotopical Algebra

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Homotopical Algebra

Daniel G. Quillen[1]

Homotopical algebra or non-linear homological algebra is the generalization of homological algebra to arbitrary categories which results by considering a simplicial object as being a generalization of a chain complex. The first step in the theory was presented in [5], [6], where the derived functors of a non-additive functor from an abelian category $\underline{A}$ with enough projectives to another category $\underline{B}$ were constructed. This construction generalizes to the case where $\underline{A}$ is a category closed under finite limits having sufficiently many projective objects, and these derived functors can be used to give a uniform definition of cohomology for universal algebras. In order to compute this cohomology for commutative rings, the author was led to consider the simplicial objects over $\underline{A}$ as forming the objects of a homotopy theory analogous to the homotopy theory of algebraic topology, then using the analogy as a source of intuition for simplicial objects. This was suggested by the theorem of Kan [10] that the homotopy theory of simplicial groups is equivalent to the homotopy theory of connected pointed spaces and by the derived category ([9], [19]) of an abelian category. The analogy turned out to be very fruitful, but there were a large number of arguments

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which were formally similar to well-known ones in algebraic topology, so it was decided to define the notion of a homotopy theory in sufficient generality to cover in a uniform way the different homotopy theories encountered. This is what is done in the present paper; applications are reserved for the future.

The following is a brief outline of the contents of this paper; for a more complete discussion see chapter introductions, Chapter I contains an axiomatic development of homotopy theory patterned on the derived category of an abelian category. In Chapter II we give various examples of homotopy theories that arise from these axioms, in particular we show that the category of simplicial objects in a category $\underline{A}$ satisfying suitable conditions gives rise to a homotopy theory. Also in §5 we give a uniform description of homology and cohomology in a homotopy theory as the "linearization" or "abelianization" of the non-linear homotopy situation, and we indicate how in the case of algebras this yields a reasonable cohomology theory.

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