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Calculus and Mechanics on Two-Point Homogenous Riemannian Spaces

Springer

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A.V. Shchepetilov, *Calculus and Mechanics on Two-Point Homogenous Riemannian Spaces*, Lect. Notes Phys. 707 (Springer, Berlin Heidelberg 2006), DOI 10.1007/b11771456

Library of Congress Control Number: 2006928280

ISSN 0075-8450
ISBN-10 3-540-35384-4 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-35384-3 Springer Berlin Heidelberg New York

Springer is a part of Springer Science+Business Media
springer.com

Printed in The Netherlands

Typesetting: by the authors and techbooks using a Springer LaTeX macro package
Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 11771456 54/techbooks 5 4 3 2 1 0

Preface

Mathematics develops both due to demands of other sciences and due to its internal logic. The latter fact explains the attention of mathematicians to many problems, which are in close connection with basic mathematical notions, even if these problems have no direct practical applications.

It is well known that the space of constant curvature is one of the basic geometry notion [208], which induced the wide field for investigations. As a result there were found numerous connections of constant curvature spaces with other branches of mathematics, for example, with integrable partial differential equations [36, 153, 189][1] and with integrable Hamiltonian systems [141]. Geodesic flows on compact surfaces of a constant negative curvature (with the genus $\geqslant 2$) generate many test examples for ergodic theory (see also [183] and the bibliography therein). The hyperbolic space $\mathbf{H}^3(\mathbb{R})$ is the space of velocities in special relativity (see Sect. 7.4.1) and also arises as space-like sections in some models of general relativity.

Long before the creation of general relativity the founders of the hyperbolic geometry, Lobachevsky and Bolyai, made initial attempts to transfer the Newtonian mechanics onto the hyperbolic space. They proposed the analog of the Newtonian potential for this space. In 1885, Killing in his important paper [87] gave a detailed consideration of the one-particle motion in the analog of the Newtonian potential on the three-dimensional sphere $\mathbf{S}^3$ and found analogs of three Kepler laws for this problem. In Liebmann papers [103] and [105], the Killing results were transferred onto the hyperbolic space $\mathbf{H}^3(\mathbb{R})$.

Soon after the appearance of quantum mechanics Schrödinger [156], Infeld, and Schild [73] considered the corresponding quantum-mechanical one-body problems for the Newtonian (Coulomb) potentials in spaces $\mathbf{S}^3$ and $\mathbf{H}^3(\mathbb{R})$ and found corresponding energy levels.

However, these results did not become widely known and at the end of the 20th century were rediscovered many times by some authors. Even the recent book on classical one-body problem in spaces $\mathbf{S}^3$ and $\mathbf{H}^3(\mathbb{R})$ [204] ignores the results of Killing and Liebmann, erroneously ascribing them to papers [37] and [93] published at the end of 20th century. Section 6.4 of present book gives

[1] The author also made his modest contribution in this subject [167].

the detailed description of the history of the one-body problem in spaces of constant curvature and thus fills the existing gap in the modern literature.

The two-body problem with central interaction in constant curvature spaces was considered firstly by the author: the classical one in [160] and the quantum mechanical one in [162]. In Euclidean space this problem is reduced to the one-body problem in a central potential after separating the center of mass motion. Due to the absence of Galilei transformations the situation for the constant curvature spaces is different [160]. The two-body problem is invariant with respect to the isometry group, but for non-Euclidean space this group is not wide enough to imply the integrability of this problem in any sense.

The natural problem of finding central potentials corresponding to integrable two-body problems is far from its solution now. At the first glance it seems a bit strange due to the diversity of methods in the theory of integrable dynamical systems, but a more closer look at these methods shows that they are not adapted for systems under consideration.

Indeed, some methods are aimed at an artificial construction of new integrable systems; other methods explain from new points of view the integrability of old integrable systems. For studying the integrability of a concrete dynamical system one can use only "old" methods: the numerical construction of Poincaré surface of section, the Painlevé test, and lucky guesses, concerning the form of additional integrals if they exist. However, in our situation the latter methods encounter great difficulties.

The numerical construction of Poincaré surfaces of section for the classical two-body system is possible only for concrete potentials of interaction. Therefore, one could not find a potential, corresponding to integrable case without a lucky guess (see also the discussion in Sect. 7.3.2). The Painlevé test (see for example [191]) practically is suitable only for differential equations having a polynomial form. It is not valid for the two-body problem in spaces of constant curvature. Lucky guesses, if appear, do it usually at once and only in relatively simple situations. In another case one should wait for the elaboration of a refined technique aimed at a concrete subject.[2]

Clearly, all reasons above, concerning the integrability of the two-body problem on constant curvature spaces, are purely heuristic and do not exhaust this problem. Note in this connection the negative results concerning the nonintegrability of the restricted [112, 213, 214] and nonrestricted [171] two-body problem in spaces of constant curvature for Newton and oscillator potentials.

The natural spaces for the two-body problem are two-point homogeneous Riemannian spaces, because a distance between bodies in these spaces is the only invariant of their location with respect to the isometry group. Simply connected spaces of a constant curvature and real projective spaces are particular cases of two-point homogeneous Riemannian spaces.[3] The two-body system in these spaces has the radial degree of freedom and degrees being described through the isometry group.

[2] Recall for example the history of the Fermat last theorem [147].

[3] Below by a two-point homogeneous space we shall mean any space of this kind except Euclidean one.

However, the problem of finding an explicitly invariant form of the two-body Hamiltonian on two-point homogeneous spaces turned out to be a difficult problem, which was solved by stages in author's papers [166, 160, 162, 163]. At last, a general formula of the two-body Hamiltonian valid for all two-point homogeneous spaces was found in [169] on the base of a special expansion of a Lie algebra $\mathfrak{g}$, corresponding to the isometry group G, into a direct sum of subspaces.

This approach uses the Helgason theory of invariant differential operators [66, 67]. In the quantum mechanical case it leads to the representation of the two-body Hamiltonian H through a radial differential operator and generators of the algebra $\mathrm{Diff}_I(Q_S)$ of invariant differential operators on the unit sphere bundle Q_S over a two-point homogeneous space Q. These generators are polynomial with respect to a base of the algebra $\mathfrak{g}$. This representation of H enables one to find separate spectral differential equations for the two-body problem on compact two-point homogeneous spaces. For Coulomb and oscillator potentials on spheres $\mathbf{S}^n$ some of these equations can be reduced to the hypergeometric equation and thus be solved in an explicit form. Therefore, Coulomb and oscillator quantum mechanical two-body problems on $\mathbf{S}^n$ are *quasi-exactly solvable* [162, 184].

Using the correspondence between classical Hamiltonian functions and quantum mechanical Hamiltonians, one can derive the explicitly invariant form of the two-body Hamiltonian function. Generators of the algebra $\mathrm{Diff}_I(Q_S)$ are replaced by generators of the corresponding graded algebra $\mathrm{gr}\,\mathrm{Diff}_I(Q_S)$, which is isomorphic to the Poisson algebra of invariant functions on the cotangent bundle T^*Q_S. Using this form of the Hamiltonian function one can compare different approaches [47, 48, 128, 152, 215] to the definition of the center of mass for two particles on constant curvature spaces. This form is also convenient for the Hamiltonian reduction of the two-body problem and for proving the absence of particles' collision on infinite time interval under some additional conditions.

Note that these investigations require various geometrical, algebraic, and analytical methods. For analyzing general situations we use here the coordinate free language, preferably in terms of corresponding Lie algebras. Indeed, for curved spaces, especially of a high dimension (for example dimension n), the existing symmetry is often hidden in cumbersome coordinate expressions. The manipulation with such expressions becomes very laborious and frequently impossible even with the help of computer algebra systems. On the other hand, expressions of invariant differential operators through base elements of a Lie algebra are polynomial with constant coefficients in non-commutative variables. Manipulations with such polynomials are much easier than coordinate evaluations.

Some results of the present book are of more general interest and can be used in other researches. These are the expression of the Laplace–Beltrami operator through a moving frame, particularly through Killing vector fields, the description of the reduced cotangent bundle over a G-homogeneous space in terms of orbits of the Ad^*_G-action, and the description of the algebra $\mathrm{Diff}_I(Q_S)$ for a two-point homogeneous space Q through generators and relations.

Chapters 1–4 describe the geometry results necessary for studying the two-body problem on two-point homogeneous spaces. The classification of these spaces are in Chap. 1. There are also models of compact two-point homogeneous spaces as submanifolds of Euclidean spaces or its factor spaces, different models of real hyperbolic spaces $\mathbf{H}^n(\mathbb{R})$, $n \geqslant 2$, and the description of the transition from compact to noncompact two-point homogeneous spaces in terms of corresponding Lie algebras.

Necessary data on differential operators are in the second chapter. Section 2.1 contains basic notions of the theory of invariant differential operators on smooth manifolds. The expression of the Laplace–Beltrami operator is derived in Sect. 2.2. Basic facts on self-adjointness of abstract differential operators and conditions sufficient for the self-adjointness of Schrödinger operators on Riemannian manifolds are described in Sect. 2.3. The last Sect. 2.4 of Chap. 2 contains the general scheme of the quantum-mechanical reduction.

The third chapter deals with algebras $\mathrm{Diff}_I(Q_S)$ of invariant differential operators on unit sphere bundles Q_S over two-point homogeneous spaces Q. There are found the description of these algebras in terms of generators and relations. All such systems of generators contain one generator D_0 of the first order. Its kernel is studied in Sect. 3.6. Also, there are found some elements from centers of these algebras.

Chapter 4 contains basic facts concerning Hamiltonian dynamical systems with symmetry and the correspondence between classical and quantum-mechanical systems. In particular, the noncommutative integrability and the momentum map are discussed here. The special symplectomorphism between a reduced cotangent bundle of a homogeneous manifold and some factor space of a coadjoint orbit of a corresponding Lie group is constructed in Sect. 4.3.4.

Chapter 5 deals with the problem of finding an explicitly invariant expression for the two-body Hamiltonian on a two-point homogeneous space Q through a radial differential operator and generators of the algebra $\mathrm{Diff}_I(Q_S)$.

The one-body problem in a central potential is discussed in Chap. 6. In Sect. 6.1 the noncommutative integrability of the classical one-body problem on a general two-point homogeneous space is proved, which seems to be a new result. For spaces of constant curvature there are given more detailed results both in classical and quantum cases. These results are known, but are collected together for the first time.

The classical case includes the discussion of the genesis of Bertrand potentials, the description of particle trajectories in these potentials, and their relations with conics in constant curvature spaces. In the quantum case there are given explicit formulas for one-particle energy levels and eigenfunctions, corresponding to Bertrand potentials in these spaces.

Also, in Sect. 6.4 there is a historical survey of one- and two-body motions in central potentials in spaces of a constant curvature, containing relevant references to many papers from the early beginning of the non-euclidean geometry till the present time.

The expression for the two-body quantum Hamiltonian from Chap. 5 is transformed into the two-body Hamiltonian function of the corresponding classical system in Chap. 7. The problem of searching of nontrivial integrals of motion for different potentials is discussed in Sect. 7.3. Also, the absence of particles' collision for some potentials and initial conditions is proved. The found expression for the two-body Hamiltonian function is considered in Sect. 7.4 with respect to the center-of-mass problem in two-point homogeneous spaces. Different existing definitions of the center-of-mass for constant curvature spaces are discussed. It is shown that all of them have flaws in comparison with the center of mass concept in Euclidean space. Reduced classical two-body systems in spaces $\mathbf{S}^n$ and $\mathbf{H}^n(\mathbb{R})$ are classified in Sect. 7.5.

Chapter 8 deals with the quantum two-body problem in compact two-point homogeneous spaces. It is shown that some infinite energy level series can be found from separate ordinary differential equations of the second order. All such equations are found for spheres $\mathbf{S}^n$; then they are reduced to the hypergeometric equation for Coulomb and oscillator potentials and corresponding energy levels series are found in explicit form. Thus, the quasi-exactly solvability of the two-body problem for Coulomb and oscillator potentials on spheres is shown. Difficulties of using this approach for noncompact two-point homogeneous spaces are discussed.

There are also four appendices in the book. The first one demonstrates the technique of calculating commutative relations for generators of differential operator algebras from Chap. 3. The second appendix contains basic facts on Fuchsian differential equations, especially on Riemann, hypergeometric, and Heun ones. In the third appendix there are some facts concerning orthogonal complex Lie algebras and their representations. Some unsolved problems arising from the book content are listed in the last appendix.

Prerequisites from differential geometry can be found in [17, 32, 34, 56, 63, 64, 92, 143, 208]; from modern theory of Hamiltonian systems in [8, 32, 58, 114, 116, 181, 193]; from the theory of Lie groups and their actions on smooth manifolds in [2, 3, 13, 31, 65, 66, 88, 134, 142, 158, 199]; from representation theory [53, 60, 135, 212]; from functional analysis in [44, 85, 144]; and many other sources.

If one is interested in a brief introduction into the one-body problem on constant curvature spaces he or she can read Sects. 1.3.3 and 6.2 for the classical case and Sects. 1.3.3, 2.3 and 6.3 for the quantum one.

The author tried to make the bibliography as complete as possible only in respect of papers, concerning one- and two-body mechanics on two-point homogeneous spaces, particularly on spaces of constant curvature, except of geodesic flows. A survey on the latter subject can be found in [25].

The author expresses his deep gratitude to A. Starinets and I. Stepanova, whose help in duration of several years made easier his access to scientific publications and understanding papers in German. He thanks A. Molev for some advices concerning representative theory, A. Sergyeyev for some useful references, P. Golubtsov for a TEXnical help, and all people who sent him their papers, cited in the book. The author is also grateful to the series editor Prof. W. Beiglböck for some critical remarks and proposals for improving the text. Any remarks and comments will be appreciated.

Moscow *Alexey V. Shchepetilov*
February 2006

Contents

Glossary

Sets

$\mathbb{N}$	is the set of natural numbers
$\mathbb{R}_+$	is the set of positive real numbers
$(\alpha, \ldots, \omega)$	denotes a set of objects $\alpha, \ldots, \omega$
pt	denotes a one point set

Spaces

$\mathcal{L}^2(M, d\nu)$	is the Hilbert space of complex-valued functions on M, square integrable w.r.t. a measure ν
$\mathcal{L}^2_{\mathrm{loc}}(M, d\nu)$	is the set of complex-valued functions on M, locally square integrable w.r.t. a measure ν
$W^{k,l}_{\mathrm{loc}}(M^n, d\mu)$	37[4]
Q	denotes a two-point homogeneous Riemannian space, different from Euclidean one
$M_{\mathbf{S}}$	denotes the unit sphere bundle over a Riemannian space M
M/G	denotes a factor space of a space M with respect to an action of a group G on it
$\mathrm{span}(e_1, \ldots, e_n)$	denotes the linear span of elements $e_1, \ldots, e_n$ from some linear space
	for a linear space L the space L^* is dual to L
	for a subspace L' of a linear space L the subspace $\mathrm{ann}\, L' \subset L^*$ is the annihilator of L', i.e., $\mathrm{ann}\, L' := (\alpha \in L^* \mid \alpha(L') = 0)$

Algebras and Groups

$\mathbb{R}$	is the field of real numbers
$\mathbb{C}$	is the field of complex numbers
$\mathbb{H}$	is the algebra of quaternions

[4] A number after notation refers to a page, where this notation is described.

$\mathbb{C}a$	is the algebra of octonions (the Cayley algebra)
$C^\infty(M)$	is the algebra of smooth real-valued functions on a smooth manifold M
$C_c^\infty(M)$	is the subalgebra of $C^\infty(M)$, consisting of functions with a compact support
$C^\infty(M, \mathbb{C})$	is the algebra of smooth complex-valued functions on M
$C_c^\infty(M, \mathbb{C})$	is the subalgebra of $C^\infty(M, \mathbb{C})$, consisting of functions with a compact support
$\mathcal{P}(T^*M)$	is the algebra of smooth real-valued functions on a cotangent bundle T^*M, polynomial on fibers
$\mathcal{X}(M)$	is the infinite-dimensional Lie algebra of smooth vector fields on M; also $\mathcal{X}(M)$ is a module over $C^\infty(M)$
$U(\mathfrak{g})$	is the universal enveloping algebra for a Lie algebra $\mathfrak{g}$
$\mathrm{LDiff}(G)$, $\mathrm{RDiff}(G)$, $\mathrm{LRDiff}(G)$	are respectively algebras of left-, right- and biinvariant differential operators on a Lie group G
$\mathrm{Diff}_G(M)$	is the algebra of G-invariant differential operators on a G-homogeneous space M
$\mathrm{LDiff}_K(G)$	is the algebra of G left-invariant and K right-invariant differential operators on G, where K is a subgroup of G
$\mathrm{O}(1,n)$, $\mathrm{O}_0(1,n)$	13
$W_{\mathcal{F}}(x)$	75
$S_p(\mathfrak{g})$	76

Operations

$\circ$	denotes the Jordan multiplication in the algebra $\mathfrak{h}_3(\mathbb{C}a)$; in other cases it denotes the composition of two operations
$\pounds_\xi T$	is the Lie derivative of a tensor field T along a vector field ξ
∇	is the Levi-Civita connection on a Riemannian manifold
$\operatorname{grad} f$	is the gradient of a function f on a Riemannian manifold
ad_X	denotes the adjoint action of an element X from a Lie algebra
Ad_q	denotes the adjoint action of an element q from a Lie group
Ad_q^*	denotes the coadjoint action of an element q from a Lie group 77
$[A, B]$	denotes the commutator in algebras, in particular the commutator of vector fields as operators, acting on functions
$\{A, B\}$	denotes the anticommutator in algebras
$[\varphi, \psi]_P$	denotes the Poisson brackets of functions φ and ψ on a Poisson manifold
$\langle \cdot, \cdot \rangle$	denotes a scalar (inner) product
$\operatorname{im} \lambda$	is the image of a map λ
λ^{-1}	is the inverse map (generally multivalued) for a map λ
id	is the identity map

π_i	denotes different projections π_1 is a projection of a group onto its homogeneous space 8, 10, 24, 97, π_2 – 33, π_3 – 54, π_4 – 98, $\tilde{\pi}_1, \tilde{\pi}_2$ – 113,
$d\pi$ and π_*	denote the differential of a map π
$d\pi^*$	denotes the codifferential of a map π
$\oplus$	denotes a direct sum of linear spaces, until indicated otherwise
$\mathrm{Kil}_{\mathfrak{g}}$	is the Killing form for a Lie algebra $\mathfrak{g}$

Miscellania

$\mathbf{i}, \mathbf{j}, \mathbf{k}$	are quaternion complex units
$\dim_{\mathbb{K}}$	denotes a dimension of some object over a field $\mathbb{K}$
$(z_1 : \ldots : z_{n+1})$	are homogeneous coordinates of a point from a projective space
$\mathrm{Re}\, z, \mathrm{Im}\, z$	are respectively real and imaginary parts of an element $z \in \mathbb{C}, \mathbb{H}, \mathbb{C}a$
$A \setminus B$	denotes the set subtracting
$\mathrm{symb}\, D$	is the symbol of a differential operator D 104
$\mathfrak{S}_l$	denotes the group of all permutations of l elements

Until indicated otherwise, all manifolds, linear spaces, algebras, etc. are supposed to be real; smooth manifolds are supposed to be Hausdorff, paracompact and second countable.

Lie groups are denoted by capital Latin letters and their Lie algebras by corresponding small gothic letters. Also, small gothic letters denote linear subspaces of Lie algebras. Four series of simple classical complex Lie algebras are denoted as $\mathfrak{A}_n, \mathfrak{B}_n, \mathfrak{C}_n, \mathfrak{D}_n$.

For a linear space V the symbol $T(V)$ denotes the tensor algebra without unit. We suppose also that the symmetric algebra $S(V)$ and the universal enveloping algebra $U(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ do not contain the unit element.

If a Lie group G acts in a linear space V, then its *invariant* means an invariant polynomial with arguments from V, i.e., an invariant element from the symmetric algebra $S(V^*)$ for the adjoint space V^*.

A scalar (inner) product in complex and quaternion linear spaces is supposed to be linear w.r.t. the second argument and conjugate linear w.r.t. the first one. A quaternion space is the right one w.r.t. quaternion multiplication.

A square root for the positive number is positive; for other numbers it is an arbitrary root.

Throughout the book by a polynomial with noncommutative arguments we mean an ordered one, i.e., each its monomial is an ordered product.

The standard abbreviations "iff" and "w.r.t." mean respectively "if and only if" and "with respect to".