Lecture Notes in Mathematics 1870

Vladimir I. Gurariy · Wolfgang Lusky

Geometry of Müntz Spaces and Related Questions

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After the completion of our book the first named author,
Vladimir I. Gurariy, died. The world lost a great mathematician
and I lost a close friend.
Wolfgang Lusky

Preface

Let $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ be an increasing sequence of non-negative numbers:

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$$

Moreover let $M(\Lambda) = \{t^{\lambda_k}\}_{k=0}^{\infty}$ be the sequence of the functions t^{λ_k} on $[0,1]$ and let $[M(\Lambda)]_E$ be the closed linear span of $M(\Lambda)$ in a given Banach space E containing $M(\Lambda)$. We call $M(\Lambda)$ a Müntz sequence and $[M(\Lambda)]_E$ a Müntz space.

In our book we shall be mainly concerned with $E = C := C[0,1]$, the Banach space of all realvalued continuous functions on $[0,1]$ endowed with the sup-norm, and $E = C_0 := C_0[0,1]$, the subspace of C consisting of all those functions $f \in C$ with $f(0) = 0$. Furthermore we deal with $E = L_p = L_p[0,1]$, $1 \leq p \leq \infty$, the space of all (classes of) realvalued measurable functions f on $[0,1]$ with

$$||f||_{L_p} = \left(\int_0^1 |f(t)|^p dt \right)^{1/p} < \infty \quad \text{if} \quad 1 \leq p < \infty \ .$$

If $p = \infty$ then we take for $||f||_{L_\infty}$ the essential sup-norm instead.

We want to study geometric properties of the corresponding Müntz sequences and spaces. Let us begin with the famous Müntz theorem, [110]:

For $E = C$ or $E = L_p$, $1 \leq p < \infty$, we have

$$[M(\Lambda)]_E \neq E \quad \text{if and only if} \quad \sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty \ .$$

(A proof of this fact in more generality will be given in 6.1.)

So, if $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$ we obtain new Banach spaces $[M(\Lambda)]_E$. This sets the stage for the central problem we discuss in (Part II of) our book:

What kind of Banach space $[M(\Lambda)]_E$ do we obtain depending on the given Λ if $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$?

This problem is far from being solved. Here we present the known theorems and prove new results in this direction. For example, if Λ is quasilacunary then $[M(\Lambda)]_{L_p}$ is isomorphic to l_p for $1 \leq p < \infty$ and $[M(\Lambda)]_C$ is isomorphic to c_0 (Sect. 9.1). But for non-quasilacunary Λ this is not always the case. There are at least two different isomorphism classes for $[M(\Lambda)]_C$ (Sect. 10.2). Moreover there is a continuum of different isometry classes for $[M(\Lambda)]_C$ (Sect. 10.4). In general, $[M(\Lambda)]_E$ can be regarded as a sequence space rather than a function space. $[M(\Lambda)]_{L_p}$ is always isomorphic to a subspace of l_p and $[M(\Lambda)]_C$ is isomorphic to a subspace of c_0 provided that the Müntz condition $\sum_k 1/\lambda_k < \infty$ and the gap condition $\inf_k(\lambda_{k+1} - \lambda_k) > 0$ are satisfied. In addition, $[M(\Lambda)]_{L_1}$ is always isomorphic to a dual Banach space (Sect. 9.1).

It is an open problem if every $[M(\Lambda)]_E$ has a basis. We discuss more general bounded approximation properties in Chap. 9. However, $[M(\Lambda)]_C$ can never have a monotone basis (Sect. 9.4). In this context it is interesting to note that $M(\Lambda)$ is always a minimal system provided that the Müntz condition holds. But $M(\Lambda)$ is never a basis or even uniformly minimal in $[M(\Lambda)]_E$ for $E = C$ or $E = L_p$ unless Λ is lacunary (Sect. 9.3). In contrast to Müntz sequences the trigonometric system $\{z^k\}_{k=-\infty}^{\infty}$ on $\{z \in \mathbf{C} \;:\; |z| = 1\}$ is uniformly minimal and even an Auerbach system. The traditional bridge between the trigonometric system and the classical Müntz system $\{t^n\}_{n=0}^{\infty}$, the substitution by Chebyshev polynomials [12], breaks down if we go over to subsequences of $\{t^n\}_{n=0}^{\infty}$. So there is no way to relate a general Müntz sequence $\{t^{\lambda_n}\}_{n=0}^{\infty}$ to the trigonometric system.

It is even unknown in general if the finite dimensional Müntz spaces $[M(\{\lambda_0, \lambda_1, \ldots, \lambda_n\})]_C$ have uniformly bounded basis constants. In Sects. 10.3 and 12.2 we discuss some special cases and related questions. In Chap. 12 we investigate phenomena which, we feel, deserve further investigation. Take a Müntz sequence $\{t^{\lambda_k}\}_{k=1}^{\infty}$, fix n and put $B_m = \operatorname{span}\{t^{\lambda_{m+1}}, \ldots, t^{\lambda_{m+n}}\}$ in C. Then, for many different $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$, the sequence of n-dimensional Banach spaces $\{B_m\}_{m=1}^{\infty}$ converges to the subspace $\operatorname{span}\{t, t\log t, \ldots, t^{n-1}\log t\}$ of C with respect to the logarithm of the Banach-Mazur distance. This might be helpful for gaining further insight in the isomorphism character of $[M(\Lambda)]_C$. In Chap. 11 we treat more general classes of subspaces of $C[0,1]$ which have many common features with $[M(\Lambda)]_C$.

It is well-known that there is a close relationship between the theory of Müntz spaces and fields like approximation theory, harmonic analysis and functional analysis. The first major contribution to this theory after the seminal papers of Müntz [110] and Szász [136] was given by L. Schwartz [128] and Clarkson and Erdös [19] who established the fact that, for integer Λ, each $x(t) \in [M(\Lambda)]_C$ has an analytic continuation to the open complex unit disk. This means, for example, that $[M(\Lambda)]_C$ consists entirely of functions which are real-analytic on $]0,1[$ provided that the Müntz condition and the gap condition hold! (See Sect. 6.2.)

In our book we want to change the accent from an analytical to a more geometrical approach and attempt to put well-known and new results into the perspective of a geometrical framework. At the same time we do not pretend completeness, we rather want to put the emphasis on unsolved problems, conjectures and ideas according to the taste of the authors. Although there is a natural overlap in this book with portions from excellent books such as [12] and [22] we present this material here from our geometric point of view. It seems to be the first time that Müntz spaces are treated under strict geometric orientation.

We assume that the reader has a basic knowledge of functional analysis.

The book is divided into two parts and twelve chapters. The first part contains the preliminary material from the geometry of normed spaces which is then applied to concrete Müntz spaces in Part II and which the authors believe to be promising for further investigation.

Both parts are essentially selfcontained and can be read independently of each other. In the summary Part I we skip some of the proofs and refer to the literature instead while, as a rule, in Part II we work out the proofs in full detail.

But Part I is more comprehensive than necessary for a simple outline of the preliminaries to Part II. There we give a systematic treatise of classical Banach space notions such as opening and inclination of subspaces (in Chap. 1). Moreover we introduce the projection function and projection type of a Banach space (1.6) and discuss their relation to Banach spaces with or without bases. Here the study of dispositions of subspaces in Banach spaces plays the main role.

In Chap. 2 we deal with general sequences in Banach spaces and properties such as minimality, completeness or stability. After the introduction of basic notions such as isomorphisms and the Banach-Mazur distance in Chap. 3 we study spaces which are (almost) universal with respect to a given class of Banach spaces and similar notions for bases in Chap. 4. Finally, Chap. 5 is devoted to a discussion of approximation properties centered around the commuting bounded approximation property (CBAP).

All our Banach spaces are assumed to be real unless indicated otherwise. (But almost all proofs in the following can be taken over literally to the complex case.) If E is a Banach space let E^* denote its topological dual space, i.e. the space of all linear bounded functionals on E.

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Contents