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Modular Forms

With 11 Figures

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Preface

Modular forms play an essential role in Number Theory. Furthermore the importance of modular forms has continued to grow in many areas of mathematics including the infinite dimensional representation theory of Lie groups and finite group theory. The aim of this book is to introduce some basic theory of modular forms of one variable.

Originally this book was written in Japanese under the title "Automorphic forms and Number Theory" by Koji Doi and myself and published by Kinokuniya, Tokyo, in 1976. When the English translation was planned, the first named author proposed that only the chapters written mainly by me be translated together with some additional material and published under my sole authorship.

In Chapters 1 and 2, the general theory of Fuchsian groups, automorphic forms and Hecke algebras is discussed. In Chapter 3, I summarize some basic results on Dirichlet series which are necessary in the succeeding chapters. In Chapter 4, the classical theories of modular groups and modular forms are studied. Here the usefulness of Hecke operators as well as the remarkable relation between modular forms and Dirichlet series obtained by Hecke and Weil have been emphasized. Chapter 5 briefly reviews quaternion algebras and their unit groups, which are also Fuchsian groups and which play a role similar to that of modular groups in their application to number theory. Chapter 6 is devoted to the trace formulae of Hecke operators by Eichler and Selberg. The formulae have been generalized by many people including H. Shimizu, H. Hijikata and H. Saito. A formula computable by them is also offered. In our Japanese edition, as an introduction to the automorphic forms of several variables, Chapter 7 deals with Eisenstein series of Hilbert modular groups and the application to values of zeta-functions (following Siegel). As a result of important series of recent work by Shimura on Eisenstein series, I decided to rewrite it to introduce some of his results on Eisenstein series restricting it to only the case of one variable.

I should like to express my deepest gratitude to Professor Goro Shimura, who constructed the essential part of the arithmetic theory of automorphic functions, for his valuable suggestions and encouragement.

The translation of Chapters 1 through 6 was prepared by my colleague Professor Yoshitaka Maeda. He also corrected mistakes in the original text, and gave me many appropriate suggestions. I express my deep and sincere thanks to him for his collaboration. I also express my hearty thanks to Professor Haruzo Hida whose lectures at Hokkaido University during 1983–84 were very helpful for the preparation of the present volume, and to Professor Hiroshi Saito and Dr. Masaru Ueda who kindly read the manuscript very carefully as a whole or in part and made many valuable suggestions.

Sapporo, February 1989 *Toshitsune Miyake*

Contents

Notation and Terminology

1. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For a rational prime p, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers, respectively. We also denote by $\mathbb{R}_+$, $\mathbb{R}_-$ and $\mathbb{C}^1$, the set of positive real numbers, the set of negative real numbers and the set of complex numbers with absolute value 1, respectively:

$$\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}, \mathbb{R}_- = \{x \in \mathbb{R} \mid x < 0\}, \mathbb{C}^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

2. For a complex number z, we denote by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, the real part and the imaginary part of z, respectively. When z is a non-zero complex number, we denote by $\arg(z)$ the argument of z, which we specify by $-\pi < \arg(z) \leqq \pi$. For a real number x, we denote by $[x]$ the largest integer not exceeding x. When x is a non-zero real number, $\mathrm{sgn}(x)$ denotes $+1$ or -1 according as $x > 0$ or $x < 0$.

3. For a ring R with unity 1, we denote by $R^\times$ the group of invertible elements in R. Further we write

$$M_n(R) = \text{the set of square matrices of degree } n \text{ over } R,$$

$$GL_n(R) = \{\alpha \in M_n(R) \mid \det(\alpha) \in R^\times\},$$

$$SL_n(R) = \{\alpha \in M_n(R) \mid \det(\alpha) = 1\}.$$

4. We denote by $\amalg$ the disjoint union of sets. For a finite set A, $|A|$ denotes the number of elements in A. We also denote by $\# \{\ldots\ldots\}$, the number of the elements of the set given by $\{\ldots\ldots\}$.

5. When $g_1, \ldots, g_m$ are elements of a group G, $\langle g_1, \ldots, g_m \rangle$ denotes the subgroup of G generated by $g_1, \ldots, g_m$. When $v_1, \ldots, v_m$ are vectors in a vector space V over a field K, $\langle v_1, \ldots, v_m \rangle$ denotes the subspace of V generated by $v_1, \ldots, v_m$. For mappings $g: A \to B$ and $f: B \to C$, we denote by $f \circ g$ the mapping of A to C given by

$$(f \circ g)(a) = f(g(a)) \qquad (a \in A).$$