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# Algebraic Theory of Locally Nilpotent Derivations

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To my wife Sheryl  
and our wonderful children,  
Jenna, Kathryn, and Ella Marie,  
whom I love very much.

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