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Tools for Computational Finance

Third Edition

Springer

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The figure in the front cover illustrates the value of an American put option. The slices are taken from the surface shown in the Figure 1.5.

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Preface to the First Edition

Basic principles underlying the transactions of financial markets are tied to probability and statistics. Accordingly it is natural that books devoted to *mathematical finance* are dominated by stochastic methods. Only in recent years, spurred by the enormous economical success of financial derivatives, a need for sophisticated computational technology has developed. For example, to price an American put, quantitative analysts have asked for the numerical solution of a free-boundary partial differential equation. Fast and accurate numerical algorithms have become essential tools to price financial derivatives and to manage portfolio risks. The required methods aggregate to the new field of *Computational Finance*. This discipline still has an aura of mysteriousness; the first specialists were sometimes called *rocket scientists*. So far, the emerging field of computational finance has hardly been discussed in the mathematical finance literature.

This book attempts to fill the gap. Basic principles of computational finance are introduced in a monograph with textbook character. The book is divided into four parts, arranged in six chapters and seven appendices. The general organization is

Part I (Chapter 1): Financial and Stochastic Background
Part II (Chapters 2, 3): Tools for Simulation
Part III (Chapters 4, 5, 6): Partial Differential Equations for Options
Part IV (Appendices A1...A7): Further Requisits and Additional Material.

The first chapter introduces fundamental concepts of financial options and of stochastic calculus. This provides the financial and stochastic background needed to follow this book. The chapter explains the terms and the functioning of standard options, and continues with a definition of the Black-Scholes market and of the principle of risk-neutral valuation. As a first computational method the simple but powerful binomial method is derived. The following parts of Chapter 1 are devoted to basic elements of stochastic analysis, including Brownian motion, stochastic integrals and Itô processes. The material is discussed only to an extent such that the remaining parts of the book can be understood. Neither a comprehensive coverage of derivative products nor an explanation of martingale concepts are provided. For such in-depth coverage of financial and stochastic topics ample references to special literature are given as hints for further study. The focus of this book is on numerical methods.

Chapter 2 addresses the computation of random numbers on digital computers. By means of congruential generators and Fibonacci generators, uniform deviates are obtained as first step. Thereupon the calculation of normally distributed numbers is explained. The chapter ends with an introduction into low-discrepancy numbers. The random numbers are the basic input to integrate stochastic differential equations, which is briefly developed in Chapter 3. From the stochastic Taylor expansion, prototypes of numerical methods are derived. The final part of Chapter 3 is concerned with Monte Carlo simulation and with an introduction into variance reduction.

The largest part of the book is devoted to the numerical solution of those partial differential equations that are derived from the Black-Scholes analysis. Chapter 4 starts from a simple partial differential equation that is obtained by applying a suitable transformation, and applies the finite-difference approach. Elementary concepts such as stability and convergence order are derived. The free boundary of American options —the optimal exercise boundary— leads to variational inequalities. Finally it is shown how options are priced with a formulation as linear complimentarity problem. Chapter 5 shows how a finite-element approach can be used instead of finite differences. Based on linear elements and a Galerkin method a formulation equivalent to that of Chapter 4 is found. Chapters 4 and 5 concentrate on standard options.

Whereas the transformation applied in Chapters 4 and 5 helps avoiding spurious phenomena, such artificial oscillations become a major issue when the transformation does not apply. This is frequently the situation with the non-standard *exotic* options. Basic computational aspects of exotic options are the topic of Chapter 6. After a short introduction into exotic options, Asian options are considered in some more detail. The discussion of numerical methods concludes with the treatment of the advanced total variation diminishing methods. Since exotic options and their computations are under rapid development, this chapter can only serve as stimulation to study a field with high future potential.

In the final part of the book, seven appendices provide material that may be known to some readers. For example, basic knowledge on stochastics and numerics is summarized in the appendices A2, A4, and A5. Other appendices include additional material that is slightly tangential to the main focus of the book. This holds for the derivation of the Black-Scholes formula (in A3) and the introduction into function spaces (A6).

Every chapter is supplied with a set of exercises, and hints on further study and relevant literature. Many examples and 52 figures illustrate phenomena and methods. The book ends with an extensive list of references.

This book is written from the perspectives of an applied mathematician. The level of mathematics in this book is tailored to readers of the advanced undergraduate level of science and engineering majors. Apart from this basic knowledge, the book is self-contained. It can be used for a course on the subject. The intended readership is interdisciplinary. The audience of this book includes professionals in financial engineering, mathematicians, and scientists of many fields.

An expository style may attract a readership ranging from graduate students to practitioners. Methods are introduced as tools for immediate application. Formulated and summarized as algorithms, a straightforward implementation in computer programs should be possible. In this way, the reader may learn by computational experiment. *Learning by calculating* will be a possible way to explore several aspects of the financial world. In some parts, this book provides an algorithmic introduction into computational finance. To keep the text readable for a wide range of readers, some of the proofs and derivations are exported to the exercises, for which frequently hints are given.

This book is based on courses I have given on computational finance since 1997, and on my earlier German textbook *Einführung in die numerische Berechnung von Finanz-Derivaten*, which Springer published in 2000. For the present English version the contents have been revised and extended significantly.

The work on this book has profited from cooperations and discussions with Alexander Kempf, Peter Kloeden, Rainer Int-Veen, Karl Riedel und Roland Seydel. I wish to express my gratitude to them and to Anita Rother, who TEXed the text. The figures were either drawn with `xfig` or plotted and designed with `gnuplot`, after extensive numerical calculations.

Additional material to this book, such as hints on exercises and colored figures and photographs, is available at the website address

`www.mi.uni-koeln.de/numerik/compfin/`

It is my hope that this book may motivate readers to perform own computational experiments, thereby exploring into a fascinating field.

Köln *Rüdiger Seydel*
February 2002

Preface to the Second Edition

This edition contains more material. The largest addition is a new section on jump processes (Section 1.9). The derivation of a related partial integro-differential equation is included in Appendix A3. More material is devoted to Monte Carlo simulation. An algorithm for the standard workhorse of inverting the normal distribution is added to Appendix A7. New figures and more exercises are intended to improve the clarity at some places. Several further references give hints on more advanced material and on important developments.

Many small changes are hoped to improve the readability of this book. Further I have made an effort to correct misprints and errors that I knew about.

A new domain is being prepared to serve the needs of the computational finance community, and to provide complementary material to this book. The address of the domain is

`www.compfin.de`

The domain is under construction; it replaces the website address `www.mi.uni-koeln.de/numerik/compfin/`.

Suggestions and remarks both on this book and on the domain are most welcome.

Köln
July 2003

Rüdiger Seydel

Preface to the Third Edition

The rapidly developing field of financial engineering has suggested extensions to the previous editions. Encouraged by the success and the friendly reception of this text, the author has thoroughly revised and updated the entire book, and has added significantly more material. The appendices were organized in a different way, and extended. In this way, more background material, more jargon and terminology are provided in an attempt to make this book more self-contained. New figures, more exercises, and better explanations improve the clarity of the book, and help bridging the gap to finance and stochastics.

The largest addition is a new section on analytic methods (Section 4.8). Here we concentrate on the interpolation approach and on the quadratic approximation. In this context, the analytic method of lines is outined. In Chapter 4, more emphasis is placed on extrapolation and the estimation of the accuracy. New sections and subsections are devoted to risk-neutrality. This includes some introducing material on topics such as the theorem of Girsanov, state-price processes, and the idea of complete markets. The analysis and geometry of early-exercise curves is discussed in more detail. In the appendix, the derivations of the Black-Scholes equation, and of a partial integro-differential equation related to jump diffusion are rewritten. An extra section introduces multidimensional Black-Scholes models. Hints on testing the quality of random-number generators are given. And again more material is devoted to Monte Carlo simulation. The integral representation of options is included as a link to quadrature methods. Finally, the references are updated and expanded.

It is my pleasure to acknowledge that the work on this edition has benefited from helpful remarks of Rainer Int-Veen, Alexander Kempf, Sebastian Quecke, Roland Seydel, and Karsten Urban.

The material of this Third Edition has been tested in courses the author gave recently in Cologne and in Singapore. Parallel to this new edition, the website `www.compfin.de` is supplied by an option calculator.

Köln — *Rüdiger Seydel*

October 2005

Contents

Notations

elements of options:

t	time
T	maturity date, time to expiration
S	price of underlying asset
	S_j, S_{ji} specific values of the price S
S_t	price of the asset at time t
K	strike price, exercise price
V	value of an option (V_C value of a call, V_P value of a put, $^\mathrm{am}$ American, $^\mathrm{eur}$ European)
σ	volatility
r	interest rate (Appendix A1)

general mathematical symbols:

$\mathbb{R}$	set of real numbers
$\mathbb{N}$	set of integers > 0
$\mathbb{Z}$	set of integers
$\in$	element in
$\subseteq$	subset of, $\subset$ strict subset
$[a, b]$	closed interval $\{x \in \mathbb{R} : a \leq x \leq b\}$
$[a, b)$	half-open interval $a \leq x < b$ (analogously $(a, b], (a, b)$)
P	probability
E	expectation (Appendix B1)
Var	variance
Cov	covariance
log	natural logarithm
$:=$	defined to be
$\doteq$	equal except for rounding errors
$\equiv$	identical
$\Longrightarrow$	implication
$\Longleftrightarrow$	equivalence
$O(h^k)$	Landau-symbol: for $h \to 0$
	$f(h) = O(h^k) \iff \frac{f(h)}{h^k}$ is bounded
$\sim \mathcal{N}(\mu, \sigma^2)$	normal distributed with expectation μ and variance σ^2
$\sim \mathcal{U}[0, 1]$	uniformly distributed on $[0, 1]$

Δt	small increment in t
^{tr}	transposed; A^{tr} is the matrix where the rows and columns of A are exchanged.
$\mathcal{C}^0[a,b]$	set of functions that are continuous on $[a,b]$
$\in \mathcal{C}^k[a,b]$	k-times continuously differentiable
$\mathcal{D}$	set in $\mathbb{R}^n$ or in the complex plane, $\bar{\mathcal{D}}$ closure of $\mathcal{D}$, $\mathcal{D}^\circ$ interior of $\mathcal{D}$
$\partial\mathcal{D}$	boundary of $\mathcal{D}$
$\mathcal{L}^2$	set of square-integrable functions
$\mathcal{H}$	Hilbert space, Sobolev space (Appendix C3)
$[0,1]^2$	unit square
Ω	sample space (in Appendix B1)
$f^+ := \max\{f, 0\}$	
$\dot{u}$	time derivative $\frac{du}{dt}$ of a function $u(t)$

integers:

$i, j, k, l, m, n, M, N, \nu$

various variables:

$X_t, X, X(t)$	random variable
W_t	Wiener process, Brownian motion (Definition 1.7)
$y(x,\tau)$	solution of a partial differential equation for (x,τ)
w	approximation of y
h	discretization grid size
φ	basis function (Chapter 5)
ψ	test function (Chapter 5)
$1_{\mathcal{D}}$	indicator function: $=1$ on $\mathcal{D}$, $=0$ elsewhere.

abbreviations:

BDF	Backward Difference Formula, see Section 4.2.1
CFL	Courant-Friedrichs-Lewy, see Section 6.5.1
Dow	Dow Jones Industrial Average
FTBS	Forward Time Backward Space, see Section 6.5.1
FTCS	Forward Time Centered Space, see Section 6.4.2
GBM	Geometric Brownian Motion, see (1.33)
MC	Monte Carlo
ODE	Ordinary Differential Equation
OTC	Over The Counter
PDE	Partial Differential Equation
PIDE	Partial Integro-Differential Equation
PSOR	Projected Successive Overrelaxation
QMC	Quasi Monte Carlo
SDE	Stochastic Differential Equation
SOR	Successive Overrelaxation

TVD	Total Variation Diminishing
i.i.d.	independent and identical distributed
inf	infimum, largest lower bound of a set of numbers
sup	supremum, least upper bound of a set of numbers
supp(f)	support of a function f: $\{x \in \mathcal{D} : f(x) \neq 0\}$

hints on the organization:

(2.6)	number of equation (2.6)
	(The first digit in all numberings refers to the chapter.)
(A4.10)	equation in Appendix A; similarly B, C, D
$\longrightarrow$	hint (for instance to an exercise)