

# New Introduction to Multiple Time Series Analysis

Helmut Lütkepohl

# New Introduction to Multiple Time Series Analysis

With 49 Figures  
and 36 Tables

 Springer

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*To Sabine*

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## Preface

When I worked on my *Introduction to Multiple Time Series Analysis* (Lütkepohl (1991)), a suitable textbook for this field was not available. Given the great importance these methods have gained in applied econometric work, it is perhaps not surprising in retrospect that the book was quite successful. Now, almost one and a half decades later the field has undergone substantial development and, therefore, the book does not cover all topics of my own courses on the subject anymore. Therefore, I started to think about a serious revision of the book when I moved to the European University Institute in Florence in 2002. Here in the lovely hills of Tuscany I had the time to think about bigger projects again and decided to prepare a substantial revision of my previous book. Because the label *Second Edition* was already used for a previous reprint of the book, I decided to modify the title and thereby hope to signal to potential readers that significant changes have been made relative to my previous multiple time series book.

Although Chapters 1–5 still contain an introduction to the vector autoregressive (VAR) methodology and their structure is largely the same as in Lütkepohl (1991), there have been some adjustments and additions, partly in response to feedback from students and colleagues. In particular, some discussion on multi-step causality and also bootstrap inference for impulse responses has been added. Moreover, the LM test for residual autocorrelation is now presented in addition to the portmanteau test and Chow tests for structural change are discussed on top of the previously considered prediction tests. When I wrote my first book on multiple time series, the cointegration revolution had just started. Hence, only one chapter was devoted to the topic. By now the related models and methods have become far more important for applied econometric work than, for example, vector autoregressive moving average (VARMA) models. Therefore, Part II (Chapters 6–8) is now entirely devoted to VAR models with cointegrated variables. The basic framework in this new part is the vector error correction model (VECM). Chapter 9 is also new. It contains a discussion of structural vector autoregressive and vector error correction models which are by now also standard tools in applied econometric

analysis. Chapter 10 on systems of dynamic simultaneous equations maintains much of the contents of the corresponding chapter in Lütkepohl (1991). Some discussion of nonstationary, integrated series has been added, however. Chapters 9 and 10 together constitute Part III. Given that the research activities devoted to VARMA models have been less important than those on cointegration, I have shifted them to Part IV (Chapters 11–15) of the new book. This part also contains a new chapter on cointegrated VARMA models (Chapter 14) and in Chapter 15 on infinite order VAR models, a section on models with cointegrated variables has been added. The last part of the new book contains three chapters on special topics related to multiple time series. One chapter deals with autoregressive conditional heteroskedasticity (Chapter 16) and is new, whereas the other two chapters on periodic models (Chapter 17) and state space models (Chapter 18) are largely taken from Lütkepohl (1991). All chapters have been adjusted to account for the new material and the new structure of the book. In some instances, also the notation has been modified. In Appendix A, some additional matrix results are presented because they are used in the new parts of the text. Also Appendix C has been expanded by sections on unit root asymptotics. These results are important in the more extensive discussion of cointegration. Moreover, the discussion of bootstrap methods in Appendix D has been revised. Generally, I have added many new references and consequently the reference list is now much longer than in the previous version. To keep the length of the book in acceptable bounds, I have also deleted some material from the previous version. For example, stationary reduced rank VAR models are just mentioned as examples of models with nonlinear parameter restrictions and not discussed in detail anymore. Reduced rank models are now more important in the context of cointegration analysis. Also the tables with example time series are not timely anymore and have been eliminated. The example time series are available from my webpage and they can also be downloaded from [www.jmulti.de](http://www.jmulti.de). It is my hope that these revisions make the book more suitable for a modern course on multiple time series analysis.

Although multiple time series analysis is applied in many disciplines, I have prepared the text with economics and business students in mind. The examples and exercises are chosen accordingly. Despite this orientation, I hope that the book will also serve multiple time series courses in other fields. It contains enough material for a one semester course on multiple time series analysis. It may also be combined with univariate times series books or with texts like Fuller (1976) or Hamilton (1994) to form the basis of a one or two semester course on univariate and multivariate time series analysis. Alternatively, it is also possible to select some of the chapters or sections for a special topic of a graduate level econometrics course. For example, Chapters 1–8 could be used for an introduction to stationary and cointegrated VARs. For students already familiar with these topics, Chapter 9 could be a special topic on structural VAR modelling in an advanced econometrics course.

The students using the book must have knowledge of matrix algebra and should also have been introduced to mathematical statistics, for instance, based on textbooks like Mood, Graybill & Boes (1974), Hogg & Craig (1978) or Rohatgi (1976). Moreover, a working knowledge of the Box-Jenkins approach and other univariate time series techniques is an advantage. Although, in principle, it may be possible to use the present text without any prior knowledge of univariate time series analysis if the instructor provides the required motivation, it is clearly an advantage to have some time series background. Also, a previous introduction to econometrics will be helpful. Matrix algebra and an introductory mathematical statistics course plus the multiple regression model are necessary prerequisites.

As the previous book, the present one is meant to be an introductory exposition. Hence, I am not striving for utmost generality. For instance, quite often I use the normality assumption although the considered results hold under more general conditions. The emphasis is on explaining the underlying ideas and not on generality. In Chapters 2–7 a number of results are proven to illustrate some of the techniques that are often used in the multiple time series arena. Most proofs may be skipped without loss of continuity. Therefore the beginning and the end of a proof are usually clearly marked. Many results are summarized in propositions for easy reference.

Exercises are given at the end of each chapter with the exception of Chapter 1. Some of the problems may be too difficult for students without a good formal training, some are just included to avoid details of proofs given in the text. In most chapters empirical exercises are provided in addition to algebraic problems. Solving the empirical problems requires the use of a computer. Matrix oriented software such as GAUSS, MATLAB, or Ox will be most helpful. Most of the empirical exercises can also be done with the easy-to-use software JMULTi (see Lütkepohl & Krätzig (2004)) which is available free of charge at the website [www.jmulti.de](http://www.jmulti.de). The data needed for the exercises are also available at that website, as mentioned earlier.

Many persons have contributed directly or indirectly to this book and I am very grateful to all of them. Many students and colleagues have commented on my earlier book on the topic. Thereby they have helped to improve the presentation and to correct errors. A number of colleagues have commented on parts of the manuscript and have been available for discussions on the topics covered. These comments and discussions have been very helpful for my own understanding of the subject and have resulted in improvements to the manuscript.

Although the persons who have contributed to the project in some way or other are too numerous to be listed here, I wish to express my special gratitude to some of them. Because some parts of the old book are still maintained, it is only fair to mention those who have helped in a special way in the preparation of that book. They include Theo Dykstra who read and commented on a large part of the manuscript during his visit in Kiel in the summer of 1990, Hans-Eggert Reimers who read the entire manuscript, suggested many

improvements, and pointed out numerous errors, Wolfgang Schneider who helped with examples and also commented on parts of the manuscript as well as Bernd Theilen who prepared the final versions of most figures, and Knut Haase and Holger Claessen who performed the computations for many of the examples. I deeply appreciate the help of all these collaborators.

Special thanks for comments on parts of the new book go to Pentti Saikkonen for helping with Part II and to Ralf Brüggemann, Helmut Herwartz, and Martin Wagner for reading Chapters 9, 16, and 18, respectively. Christian Kascha prepared some of the new figures and my wife Sabine helped with the preparation of the author index. Of course, I assume full responsibility for any remaining errors, in particular, as I have keyboarded large parts of the manuscript myself. A preliminary L<sup>A</sup>T<sub>E</sub>X version of parts of the old book was provided by Springer-Verlag. I thank Martina Bihn for taking charge of the project on the side of Springer-Verlag. Needless to say, I welcome any comments by readers.

Florence and Berlin,

*Helmut Lütkepohl*  
March 2005



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