Encyclopaedia of Mathematical Sciences
Volume 133

Invariant Theory and Algebraic Transformation Groups IV

Subseries Editors:
R.V. Gamkrelidze V.L. Popov

E. A. Tevelev

Projective Duality and Homogeneous Spaces

Springer

Author

Evgueni A. Tevelev
Department of Mathematics
University of Texas at Austin
78712 Austin, Texas, USA
e-mail: tevelev@math.utexas.edu

Founding editor of the Encyclopaedia of Mathematical Sciences:
R. V. Gamkrelidze

Mathematics Subject Classification (2000): 14-XX

ISSN 0938-0396
ISBN 3-540-22898-5 Springer Berlin Heidelberg New York

Typesetting: by the authors using a Springer LaTeX macro package
Production: LE-TeX Jelonek, Schmidt & Vöckler GbR, Leipzig
Cover Design: E. Kirchner, Heidelberg, Germany
Printed on acid-free paper 46/3142 YL 5 4 3 2 1 0

Preface

During several centuries various reincarnations of projective duality have inspired research in algebraic and differential geometry, classical mechanics, invariant theory, combinatorics, etc. To put it simply, projective duality is the systematic way of recovering a projective variety from the set of its tangent hyperplanes. In this survey I have tried to emphasize that there are many different aspects of projective duality and that it can be studied using a wide range of methods. But at the same time I was pushing hard to minimize the technical details in the hope of writing a text that requires a knowledge of only basic facts from algebraic geometry and the theory of algebraic (or Lie) groups. Most proofs in this book are compilations from various sources.

Projective duality is defined for arbitrary projective varieties and it does not seem natural a priori to consider varieties with symmetries. However, it turns out that many important examples carry the natural action of the algebraic group. This is especially true for projective varieties that have extremal projective properties: self-dual varieties, varieties of positive defect, Severi varieties, Scorza varieties, varieties of small codegree, etc. I have tried to emphasize this phenomenon in this survey. However, one aspect is totally omitted – I decided against including results about dual varieties of toric varieties and A-discriminants. This theory is presented in the beautiful book by Gelfand, Zelevinsky, and Kapranov [GKZ2] (the author always keeps it under his pillow) and I don't feel I have anything to add to it.

Let us say a few words about the contents of this survey.

Chapter 1 is intended as a brief introduction to projective duality. All results here were already well-known in the 19th century. After giving basic definitions in Sect. 1.1 we discuss duality for plane curves in Sect. 1.2. We give parametric equations for dual curves, discuss the connection to the Legendre transform, introduce Plücker formulas, and study curves of degree 2, 3, and 4. In Sect. 1.3 we prove the Reflexivity Theorem. Here we follow the exposition in [GKZ2] and use the interpretation of a conormal variety as a Lagrangian subvariety in the cotangent bundle. We introduce the discriminant as a defining equation for the dual variety and the defect that measures how far the dual variety is from being a hypersurface. We prove that if the defect is positive then the variety is ruled. Finally, in Sect. 1.4 we study the behaviour of a dual variety under projections and introduce the standard notation of algebraic geometry related to divisors and line bundles.

In our survey we are mostly interested in the study of varieties with symmetries, and in Chap. 2 we study projective geometry of homogeneous spaces; more precisely we look at orbit closures for algebraic groups acting on projective spaces with finitely many orbits. In Sect. 2.1 we give the necessary background on algebraic groups and fix notation. In Sect. 2.2 we discuss Pyasetskii pairing, which is an interesting instance of projective duality. We give some standard and a few exotic examples. The next Sect. 2.3 contains the systematic treatment of actions related to gradings of simple Lie algebras. These actions provide a wealth of very important varieties that will be studied throughout this book. Some examples include Severi varieties, smooth self-dual varieties, smooth building blocks for varieties of positive defect, varieties of small codegree, etc. It is quite an interesting phenomenon that varieties with extremal projective properties tend to have maximal symmetries. We finish this chapter with the description of Pyasetskii pairing for actions related to gradings of GL_n called the Zelevinsky involution.

In Chap. 3 we study projectively dual varieties using calculations in local coordinates. In Sect. 3.1 we prove a formula due to Katz that expresses the dimension of a dual variety in terms of the hessian of local equations of a variety. We use it to prove a formula of Weyman and Zelevinsky that expresses the defect of a Segre embedding of a product of two varieties. In Sect. 3.2 we introduce a gadget called the second fundamental form that incorporates these calculations. We prove some results of Griffiths and Harris about the second fundamental form. We finish this section with the description of higher fundamental forms of flag varieties obtained by Landsberg.

In Chap. 4 we study projective constructions related to projective duality but also having a merit of their own. We prove a theorem of Zak and Ran that the Gauss map of a smooth variety is a normalization. We introduce secant and tangent varieties in Sect. 4.2, prove the Terracini Lemma, give a method for the calculation of multisecant varieties of homogeneous spaces, discuss the relationship discovered by Zak between the degree of a dual variety and the order of the variety, and give an overview of old and new results related to the Waring problem for forms. In Sect. 4.3 we discuss theorems of Zak related to the Hartshorne conjecture – theorems on tangencies, on linear normality and on Severi varieties. The main tool is a connectedness theorem of Fulton and Hansen. We finish in Sect. 4.4 by explaining the Cayley trick for Chow forms.

The dual variety is the image of the conormal variety which is the projectivized conormal bundle if the variety is smooth. In Chap. 5 we exploit this relation between duality and vector bundles. In Sect. 5.1 we prove a theorem of Holme and Ein about the defect of a smooth effective very ample divisor. We deduce this result from a theorem of Munoz about the dimension of the linear span of a tangential variety. We discuss the related notion of projective extendability. In Sect. 5.2 we apply Hartshorne's ample vector bundles to prove a theorem of Ein that a dual variety of a smooth complete intersection is a hypersurface. We introduce resultants and prove a classical theorem that they are well-defined. We also explain the Cayley trick for resultants. In the last Sect. 5.3 we describe the "Cayley method" developed by Gelfand, Kapranov, and Zelevinsky. The idea is to show that the dual variety is represented in the derived category by Koszul complexes of jet bundles. The discriminant is then equal to the "Cayley determinant" of a generically exact complex. As an application we deduce some classical formulas for discriminants and their degrees.

In Chap. 6 we discuss about the degree of dual varieties and resultants. We start in Sect. 6.1 by recalling Chern classes and then proving a formula of Katz, Kleiman, and Holme that expresses the degree of a dual variety in terms of Chern classes of the cotangent bundle. We give many examples and generalizations. We prove a theorem of De Concini and Weyman about the formula for the degree with non-negative coefficients. In Sect. 6.2 we discuss formulas for the codegree and ranks related to the Cayley method, such as a formula due to Lascoux.

In Chap. 7 we study varieties with positive defect. In Sect. 7.1 we focus on beautiful theorems of Ein about the normal bundle of a generic contact locus. Since this locus is a projective subspace, it is possible to use the machinery of vector bundles on projective spaces. We prove Ein's theorem that this normal bundle is symmetric and uniform, which explains among other things a parity theorem of Landman. We introduce the Beilinson spectral sequence and use it to calculate the normal bundle to a generic contact locus in small dimensions. Finally, we study dual varieties of scrolls and prove a theorem of Ein that a variety of defect at least 2 is a scroll if and only if the normal bundle to a generic contact locus splits. In Sect. 7.2 we follow [IL] and discuss linear systems of quadrics of constant rank and how they are related to dual varieties via the second fundamental form. In Sect. 7.3 we prove a theorem of Beltrametti, Fania, and Sommese that relates the defect of a projective variety and its Mori-theoretic characteristic called the nef value. We give a brief survey of necessary results from Mori theory. We finish by giving a classification of smooth varieties of positive defect up to dimension 10 obtained by many authors and initiated by Ein. Finally, in Sect. 7.4 we use this connection with Mori theory to classify all flag varieties with positive defect. This approach was developed by Snow in contrast with the original proof of Knop and Menzel that used the Katz dimension formula.

In Chap. 8 we study dual varieties and discriminants of several special homogeneous spaces. We start in Sect. 8.1 by showing how to use standard results of representation theory such as the Borel–Weil–Bott theorem, the BGG homomorphism, identities with Schur functors, and formulas of the Schubert calculus to find the codegree of Grassmannians or full and partial flag varieties. We give a list of formulas for the degree of hyperdeterminants and sketch the proof of a theorem of Zak about varieties of codegree 3. In Sect. 8.2 we generalize the theorem of Matsumura and Monsky about automorphisms of smooth hypersurfaces to automorphisms of smooth very ample divisors on flag varieties. In Sect. 8.3 we study commutative algebras without identities from the "discriminantal" point of view. As a corollary we prove that the algebra of diagonal matrices does not have quasiderivations. In Sect. 8.4 we study anticommutative algebras (nets of skew-symmetric forms). We show that they have beautiful geometric properties related to cubic surfaces, Del Pezzo surfaces, representation theory of S_5, etc. In Sect. 8.5 we show that the discriminant in a simple Lie algebra defined by analogy with the disciminant of a linear operator is equal to the discriminant of the minimal orbit, the so-called adjoint variety. Finally, in Sect. 8.6 we study related questions about schemes of zeros of irreducible homogeneous vector bundles. In particular, we address a question of classifying irreducible homogeneous vector bundles with a trivial line subbundle, find the maximal dimension of an isotropic subspace of a generic symmetric or skew-symmetric form, and study properties of the related Moore–Penrose involution.

In Chap. 9 we study self-dual varieties, i.e. varieties isomorphic to their projectively dual variety. In Sect. 9.1 we consider smooth self-dual varieties. The complete list of these varieties is (conjecturally) surprisingly short. All known varieties are flag varieties so we start by considering this case, where everything follows from the classification of flag varieties of positive defect. After a brief introduction to the Hartshorne conjecture we sketch the proof of the amazing theorem of Ein that gives the complete list of self-dual varieties in the range that is allowed by the Hartshorne conjecture. We also prove a finiteness theorem of Muñoz that uses the distribution of primes to give restrictions on the Beilinson spectral sequence. We finish in Sect. 9.2 by describing results of Popov about self-dual nilpotent orbits.

In the final Chap. 10 we study how the topology of the variety is reflected in singularities of the dual variety. We start in Sect. 10.1 by proving the class formula and its variant due to Landman that relates the degree of the dual variety and the Euler characteristic of the variety and its hyperplane sections. In the singular case this formula was proved by Ernström, but the Euler characteristic has to be substituted by the degree of the Chern–Mather class. In Sect. 10.2 we prove theorems of Dimca, Nemethi, Aluffi and others that multiplicities of the dual variety are given by Milnor numbers (or classes). To give an example we follow Aluffi and Cukierman and calculate multiplicities of the dual variety to a smooth surface. Finally, we give some results of Weyman and Zelevinsky about singularities of hyperdeterminants.

I would like to thank F. Zak for very inspiring discussions of projective geometry and R. Muñoz for his insights about positive defect varieties. I am grateful to S. Keel, V. Popov, A. Kuznetsov, P. Katsylo, D. Saltman, E. Vinberg for many helpful remarks and encouragement. P. Aluffi, F. Cukierman, N.C. Leung, A.J. Sommese and many others sent me lots of comments about the preliminary version of this survey.

Parts of this book were written during my visits to the Erwin Schrödinger Institute in Vienna, Mathematical Institute in Basel, and University of Glasgow. I am grateful to my hosts for the warm hospitality.

Moscow–Edinburgh–Austin, *Jenia Tevelev*
August 2004

Contents

1 Introduction to Projective Duality ... 1
1.1 Projectively Dual Varieties ... 1
1.2 Dual Plane Curves ... 2
1.2.1 Parametric Equations ... 2
1.2.2 Legendre Transformation ... 3
1.2.3 Plücker Formulas ... 4
1.2.4 Curves of Small Degree ... 5
1.3 Reflexivity Theorem ... 6
1.3.1 Proof of the Relexivity Theorem ... 6
1.3.2 Defect and Discriminant ... 9
1.4 Projections and Linear Normality ... 11
1.4.1 Projections ... 11
1.4.2 Degenerate Varieties ... 12
1.4.3 Linear Normality ... 13

2 Actions with Finitely Many Orbits ... 17
2.1 Algebraic Groups ... 17
2.2 Pyasetskii Pairing and Kashin Examples ... 32
2.3 Actions Related to Gradings ... 35
2.3.1 Construction ... 35
2.3.2 Short Gradings ... 41
2.3.3 Multisegment Duality ... 49

3 Local Calculations ... 57
3.1 Calculations in Coordinates ... 57
3.1.1 Katz Dimension Formula ... 57
3.1.2 Defect of a Product ... 60
3.2 Fundamental Forms ... 64
3.2.1 Second Fundamental Form ... 64
3.2.2 Higher Fundamental Forms ... 67
3.2.3 Fundamental Forms of Flag Varieties ... 70

4 **Projective Constructions** ... 73
4.1 Gauss Map ... 73
4.2 Tangents and Secants ... 74
4.2.1 Terracini Lemma ... 74
4.2.2 Multisecant Varieties of Homogeneous Spaces ... 75
4.2.3 $\deg X^*$ and $\operatorname{ord} X$ ... 77
4.2.4 Waring Problem for Forms ... 79
4.3 Zak Theorems ... 80
4.3.1 Theorem on Tangencies ... 80
4.3.2 Theorem on Linear Normality ... 82
4.3.3 Theorem on Severi Varieties ... 83
4.3.4 Connectedness Theorem of Fulton and Hansen ... 84
4.4 Chow Forms ... 87

5 **Vector Bundles Methods** ... 89
5.1 Dual Varieties of Smooth Divisors ... 89
5.1.1 Linear Envelope of a Tangential Variety ... 89
5.1.2 Dual Varieties of Smooth Divisors ... 91
5.1.3 Projective Extendability ... 93
5.2 Ample Vector Bundles ... 94
5.2.1 Definitions ... 94
5.2.2 Dual Varieties of Smooth Complete Intersections ... 95
5.2.3 Resultants ... 96
5.3 Cayley Method ... 97
5.3.1 Jet Bundles and Koszul Complexes ... 97
5.3.2 Cayley Determinants of Exact Complexes ... 99
5.3.3 Discriminant Complexes ... 102
5.3.4 Cayley Method for Resultants ... 105

6 **Degree of the Dual Variety** ... 109
6.1 Katz–Kleiman–Holme Formula ... 109
6.1.1 Chern Classes ... 109
6.1.2 Top Chern Class of the Jet Bundle ... 110
6.1.3 Formulas with Positive Coefficients ... 113
6.1.4 Degree of the Resultant ... 114
6.2 Formulas Related to the Cayley Method ... 115
6.2.1 Degree of the Discrminant ... 115
6.2.2 Lascoux Formula ... 116

7 **Varieties with Positive Defect** ... 119
7.1 Normal Bundle of the Contact Locus ... 119
7.1.1 Ein Theorems ... 119
7.1.2 Monotonicity Theorem ... 124
7.1.3 Beilinson Spectral Sequence ... 124
7.1.4 Planes in the Contact Locus ... 126

7.1.5 Scrolls ... 129
7.2 Linear Systems of Quadrics of Constant Rank ... 130
7.3 Defect and Nef Value ... 136
7.3.1 Some Results from Mori Theory ... 136
7.3.2 The Nef Value and the Defect ... 140
7.3.3 Varieties with Small Dual Varieties ... 145
7.4 Flag Varieties with Positive Defect ... 147
7.4.1 Nef Cone of a Flag Variety ... 147
7.4.2 Nef Values of Flag Varieties ... 149
7.4.3 Flag Varieties of Positive Defect ... 150

8 Dual Varieties of Homogeneous Spaces ... 155
8.1 Calculations of $\deg X^*$ ... 155
8.1.1 Borel–Weyl–Bott Theorem ... 155
8.1.2 Representation Theory of GL_n ... 157
8.1.3 Dual Variety of the Grassmannian ... 158
8.1.4 Codegree of G/B ... 159
8.1.5 A Closed Formula ... 161
8.1.6 Degree of Hyperdeterminants ... 166
8.1.7 Varieties of Small Codegree ... 167
8.2 Matsumura–Monsky Theorem ... 169
8.3 Discriminants of Commutative Algebras ... 171
8.3.1 Commutative Algebras Without Identities ... 171
8.3.2 Quasiderivations ... 172
8.4 Discriminants of Anticommutative Algebras ... 174
8.4.1 Generic Anticommutative Algebras ... 174
8.4.2 Regular Algebras ... 179
8.4.3 Regular 4-dimensional Anticommutative Algebras ... 181
8.4.4 Dodecahedral Section ... 183
8.5 Adjoint Varieties ... 186
8.6 Homogeneous Vector Bundles ... 189
8.6.1 Zeros of Generic Global Sections ... 189
8.6.2 Isotropic Subspaces of Forms ... 192
8.6.3 Moore–Penrose Inverse and Applications ... 196

9 Self-dual Varieties ... 207
9.1 Smooth Self-dual Varieties ... 207
9.1.1 Self-dual Flag Varieties ... 207
9.1.2 Hartshorne Conjecture ... 209
9.1.3 Ein's Theorem ... 210
9.1.4 Finiteness Theorem ... 212
9.2 Self-Dual Nilpotent Orbits ... 213

10 Singularities of Dual Varieties 219
10.1 Class Formula 219
10.2 Singularities of X^* 222
10.2.1 Milnor Numbers 222
10.2.2 Milnor Class 224
10.2.3 Dual Variety of a Surface 228
10.2.4 Singularities of Hyperdeterminants 231

References 233

Index 245