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Locally Convex Spaces

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To Szazy

Preface

The present book grew out of several courses which I have taught at the University of Zürich and at the University of Maryland during the past seven years. It is primarily intended to be a systematic text on locally convex spaces at the level of a student who has some familiarity with general topology and basic measure theory. However, since much of the material is of fairly recent origin and partly appears here for the first time in a book, and also since some well-known material has been given a not so well-known treatment, I hope that this book might prove useful even to more advanced readers. And in addition I hope that the selection of material marks a sufficient set-off from the treatments in e.g. N. Bourbaki [4], [5], R.E. Edwards [1], K. Floret-J. Wloka [1], H.G. Garnir-M. De Wilde-J. Schmets [1], A. Grothendieck [13], H. Heuser [1], J. Horváth [1], J. L. Kelley-I. Namioka et al. [1], G. Köthe [7], [10], A. P. Robertson-W. Robertson [1], W. Rudin [2], H.H. Schaefer [1], F. Trèves [1], A. Wilansky [1].

A few sentences should be said about the organization of the book. It consists of 21 chapters which are grouped into three parts. Each chapter splits into several sections. Chapters, sections, and the statements therein are enumerated in consecutive fashion. Cross references are usually k.m.n. meaning that reference is made to statement n of section m in chapter k. Inside of k.m (i.e. of section m in chapter k) reference is simply to n rather than to k.m.n. Definitions are not listed as separate statements, they are only distinguished from the text by using spaced type characters. The end of a proof is marked by ■.

Starting from Chapter 2, each chapter concludes with a reference section, intended to give an (admittedly vague) impression on the origin of several results presented in the foregoing sections, to provide furtheron a certain amount of additional information, and also to suggest further reading. It should be mentioned, however, that despite its extent, and even with regard to only those topics which are covered in this book, the corresponding bibliography is far from being complete.

Let us continue by looking very briefly at the content of the book. The first part, consisting of chapters 1–7, is devoted to the elementary theory of general topological vector spaces. Chapter 1 presents a certain amount of basic material on linear spaces of arbitrary dimension, Chapter 2 is concerned with the most fundamental concepts on topological vector spaces, Chapter 3 discusses various completeness concepts, and Chapter 4 deals with inductive linear topologies, in particular with quotients, direct sums, and inductive limits. One of the most important principles, the closed graph

theorem, is treated together with its relatives in Chapter 5 from a rather general point of view, within the setting of the webbed spaces of De Wilde and Robertson.

Locally convex spaces make their first appearance in Chapter 6. Their elementary theory is even developed within the frame of locally r -convex spaces, $0 < r \leq 1$. Chapter 7 then already lives from these concepts. The Hahn-Banach theorem is proved together with its most important consequences, a version of the Krein-Milman theorem for locally r -convex spaces is presented, and the Riesz representation theorem is derived for finite regular Borel measures on an arbitrary completely regular space.

The validity of the Hahn-Banach theorem is what makes locally convex spaces superior to other classes of topological vector spaces. It assures the existence of sufficiently many continuous linear forms to support a powerful duality theory. This theory is the central theme of the second part of the book. Its fundamental aspects are discussed in Chapter 8. In Chapter 9, the influence of considering the equicontinuous sets in the dual of a locally convex space as an inductive family of compact spaces is examined. Among others, Grothendieck's completeness theorem, the Banach-Dieudonné theorem, and Pták's results on B-completeness are derived, and the theorems of Eberlein, Krein, and Šmulian are proved in substantial generality.

Chapter 10 presents an introduction to the theory of Schwartz spaces through local (or Mackey) convergence, which does not follow the known tracks; instead, it seems to be more effective than the usual approaches. Further topics are diametral dimensions and power series spaces. In Chapter 11 some additional important classes of locally convex spaces are investigated, such as barrelled spaces, quasi-barrelled spaces, reflexive and semi-reflexive spaces, Montel and semi-Montel spaces. The basic material on uniformly convex Banach spaces and on Hilbert spaces is included for the purpose of later reference and in order to be self-contained.

The unifying theme of Chapter 12 is dual characterization of complete metrizability for locally convex spaces, and related questions. It is based on Grothendieck's theory of DF -spaces, but it extends this theory by inclusion of several recent results on certain weakenings of barrelledness, on fundamental sequences of bounded sets, and by clarifying the relationship to Schwartz topologies. Chapter 13 is devoted to a fairly general development of the theory of bornological and ultrabornological spaces. Among others, it contains the theorems of Mackey-Ulam and of Nachbin-Shirota with full proofs.

Topological bases and related objects are investigated in some detail in Chapter 14. Among the topics treated in that chapter are the continuity theorem, the weak basis theorem, shrinking and boundedly complete bases, the Orlicz-Pettis theorem, unconditional bases, and absolute bases.

The final part of the book is concerned with tensor products of locally convex spaces, various spaces of operators, and related classes of locally convex spaces, in particular with nuclear locally convex spaces. Tensor products make their natural appearance when trying to obtain results on bilinear mappings by using results on linear mappings. The tensor product in the category of locally convex spaces is Grothendieck's projective tensor product which is investigated in Chapter 15. In a sense dual to this is

the notion of the injective tensor product of locally convex spaces which is discussed in Chapter 16.

Chapter 17 is devoted to some classical spaces of continuous operators between locally convex spaces, in particular between Banach spaces, and their relations to tensor products. Chapter 18 comprises the basic results on spaces with approximation property. The theory of ideals of operators in Banach spaces, in the sense of Pietsch, is worked out in some detail in Chapter 19. Some particular aspects of this theory in presence of Hilbert spaces are discussed in Chapter 20, by using Grothendieck's inequality. The study of ideals leads to a rather rapid development of the theory of nuclear spaces in Chapter 21. Besides giving various characterizations of such spaces e.g. through ideals, tensor products, summability conditions, diametral dimensions, Choquet simplexes, special attention is paid to co-nuclear spaces, strongly nuclear spaces, and to Schauder bases in nuclear spaces.

In the preceding lines I have tried to give some rough information on what is contained in this book. Much more could be said about what is not contained in it. Some specific omissions may even be considered serious by other specialists. I am surely aware of this. But I was unfortunately not quite free concerning the extent of this book, and furtheron it was not my intention to write an encyclopedic treatise on the subject.

It was already mentioned that a prerequisite for a successful reading of the book is familiarity with the basic facts from general topology (without uniform spaces) and measure theory. These facts will not be rederived here. More than what will be needed is contained in N. Bourbaki [2], J. Dugundji [1], J.L. Kelley [1], or H. Schubert [1] (topology), and in J.J. Benedetto [1], P.R. Halmos [1], or W. Rudin [1] (measure theory). Convergence in topological spaces will be described by filters (N. Bourbaki [2]) or by nets (J.L. Kelley [1]), according to whatever seems to be more convenient. If a reader feels uncertain about some set-theoretic manipulations, then he is recommended to read P.R. Halmos [2].

The necessary background on general linear spaces will be prepared in the first chapter; this will also provide an opportunity to fix some notation. Otherwise, the notation and terminology employed here will be standard and need no extra explanation. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ will always be used to denote the sets of all natural numbers (= positive integers), of all integers, and of all rational numbers, respectively. The symbol $\mathbb{K}$ is reserved to denote either the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers. In addition, $\mathbb{N}_0$, $\mathbb{R}_+$, $\mathbb{R}_*$ will be used to denote the respective sets $\mathbb{N} \cup \{0\}$, $\{r \in \mathbb{R} \mid r \geq 0\}$, and $\{r \in \mathbb{R} \mid r > 0\}$. Occasionally, it will be convenient to call "rational numbers" not only the elements of $\mathbb{Q}$ but also those of $\mathbb{Q} \times \mathbb{Q} \subset \mathbb{C}$.

In preparation of the present book, I have received valuable suggestions from many colleagues. It is a great pleasure to record my gratitude to them at this place. In particular, I am grateful to G. Köthe, R. Ott, and K.D. Schmidt for reading parts of the manuscript and for their helpful comments, to E. Dubinsky, A. Pietsch, and W. Ruess for several fruitful discussions, to M. A. Kaashoeck for a stimulating idea, and to W.L. Hill and E.W. Strauli for linguistic assistance. I am also indebted to the publishers for their care and cooperation.

8 Preface

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Zürich, Spring 1980

H. Jarchow

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