Rolf-Peter Holzapfel

Ball and Surface Arithmetics

Aspects of Mathematics

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Ball and Surface Arithmetics

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Preface

This monograph is based on the work of the author on surface theory connected with ball uniformizations and arithmetic ball lattices during several years appearing in a lot of special articles. The first four chapters present the heart of this work in a self-contained manner (up to well-known basic facts) increased by the new functorial concept of orbital heights living on orbital surfaces. It is extended in chapter 6 to an explicit HURWITZ theory for CHERN numbers of complex algebraic surfaces with the mildest singularities, which are necessary for general application and proofs. The chapter 5 is dedicated to the application of results in earlier chapters to rough and fine classifications of PICARD modular surfaces. For this part we need additionally the arithmetic work of FEUSTEL whose final results are presented without proofs but with complete references. We had helpful connections with Russian mathematicians around VENKOV, VINBERG, MANIN, SHAFAREVICH and the nice guide line of investigations of HILBERT modular surfaces started by HIRZEBRUCH in Bonn. More recently, we can refer to the independent (until now) study of Zeta functions of PICARD modular surfaces in the book [L-R] edited by LANGLANDS and RAMAKRISHNAN. The basic idea of introducing arrangements on surfaces comes from the monograph [BHH], (BARTHEL, HÖFER, HIRZEBRUCH) where linear arrangements on the complex projective plane $\mathbb{P}^2$ play the main role. Our monograph can be understood as a broad generalization of results presented in [BHH] for the important special case of (locally) finite coverings of the basic surface $\mathbb{P}^2$ whose branch loci consist of linear arrangements. Together with [HO4] our book is part of a trilogy. Our first book [HO4] gives relations with special systems of algebraic differential equations of PICARD-FUCHS-GAUSS-MANIN type. The second book [HO8] studies mainly special values of certain PICARD modular forms related with class fields and transcendence theory. I have to thank Mrs. B. Wüst, Mrs. D. Protzek, Mr. U. Bellack for their successful fight with new typing technics, and also Ms. A. Hegewald for her skillful setting of approx. 300 diagrams and drawings.

Berlin, January 1998

R.-P. Holzapfel

Introduction

The construction of (compact real) RIEMANN surfaces C is the starting point of fruitful ideas in mathematics. On the one hand, each such surface appears as finite covering of the compactified complex GAUSS plane $\mathbb{P}^1$, on the other hand it is a compactified quotient $\mathbb{D}/\Gamma$ of the complex unit disc $\mathbb{D}$ by a suitable discrete subgroup Γ (non-euclidean lattice) of the analytic automorphism group of $\mathbb{D}$. For the classification of RIEMANN surfaces the genus or equivalently, the EULER number, is the most important invariant. It connects algebraic properties, analytic integrals, metrics and topology (number of holes) with each other. For the calculation of this invariant the HURWITZ genus formula is most important. This formula reduces the genus calculation of a finite $\mathbb{P}^1$ -covering to the determination of points and indices of ramification. The genus of the algebraic compactification of $\mathbb{D}/\Gamma$ is closely connected with a non-euclidean integral on a fundamental domain of Γ on the disc $\mathbb{D}$.

In higher dimensions the RIEMANN-ROCH theory is the most important tool for connecting analytic, algebraic and topological properties of manifolds. The monograph is concentrated to the complex dimension 2, more precisely to complex algebraic surfaces. In the smooth compact case there are two basic invariants: the EULER number c_2 and the signature. They are connected with arithmetic genus and canonical selfintersection c_1^2 of a canonical divisor by elementary formulas (proved by non-elementary RIEMANN-ROCH theory). Both kinds of constructions of RIEMANN surfaces described above, work also in higher dimensions, especially in complex dimension 2. The basic surface for finite coverings is the complex projective plane $\mathbb{P}^2$. Via general projections it is clear that each projective surface is a finite cover of $\mathbb{P}^2$.

The most natural domain generalizing $\mathbb{D}$ for the quotient construction is the complex unit ball $\mathbb{B}$. As symmetric domain it has a nice invariant hermitian metric well-known to differential geometers as BERGMANN metric. It has negative constant sectional curvature. Quotient surfaces $\mathbb{B}/\Gamma'$, Γ' a neat ball lattice, inherit this nice EINSTEIN-KÄHLER metric immediately in obvious manner. These are very special quasi-projective surfaces. Admitting also ball lattices Γ , which are not neat, one gets a much greater class of quasi-projective surfaces. Together with their compactifications we call them ball quotient surfaces. It is a working hypothesis or philosophy of the author that, up to birational equivalence and compactifications, all complex algebraic surfaces are ball quotients. No contradicting argument is known, at least to the author. On the other hand it is a fact that

Theorem 0.0.1 *On each projective complex surface there exists a ZARISKI-open part supporting an EINSTEIN-KÄHLER metric with negative constant sectional curvature coming from the ball.*

Proof Namely, we know that the complex projective plane $\mathbb{P}^2$ is a ball quotient surface. The branch locus of the corresponding locally finite covering $\mathbb{B} \rightarrow \mathbb{P}^2 - \{4 \text{ points}\}$ consists of 6 projective lines (minus the 4 points), see Proposition 5.1.3. The complement of the 6 lines inherits the ball metric. Omitting branch and ramification loci this metric can be lifted to a ZARISKI-open part of each finite cover of $\mathbb{P}^2$. □

Our point of view emphasizes the important role of finite coverings in surface theory. It is also clear that the philosophy forces to a productive work with non-smooth algebraic surfaces. The main role play quotient and (ball) cusp singularities. The latter come from compactifications of non-compact ball quotient surfaces, by definition. For a finite covering $f : X \rightarrow Y$ of two quasi-smooth surfaces it is not possible in general to resolve simultaneously the (quotient) singularities of X and Y preserving the finite covering relation. But with the trick of GALOIS closure it is not difficult to see that we can shrink the class of singularities by suitable simultaneous modifications. On this way our numerical studies of finite coverings can be reduced to surfaces with at most HIRZEBRUCH-JUNG (or cyclic) singularities and to GALOIS coverings.

The main purpose of the monograph is the introduction of special rational invariants, called orbital heights, and to work with them producing several interesting results. These heights are postulated to be invariant with respect to finite coverings up to a well-determined factor. The idea does not work for surfaces in the usual sense. A finer understanding is necessary. For this purpose orbital surfaces are introduced. These are complex surfaces together with an arrangement. An arrangement is a potentially assumed branch locus whose components consisting of (singular) points and irreducible curves are endowed with natural weights and rational coefficients. The only use of cycles instead seems, unfortunately, not fine enough for our aims. In order to find such heights of orbital surfaces one has to prove a lot of functorial properties going inductively through the categories of abelian points, orbital curves, orbital surfaces and the corresponding categories of relative objects. The diagram (D.4.3) in chapter 6 connects the functorial properties of several local and global invariants needed as contributions for orbital heights. It connects GALOIS theory, a fine arithmetic of singularities, weight shifting, reductions, localizations, base changes, additive and

multiplicative properties with each other, going through all six categories just mentioned. This fundamental diagram is delegated to the last chapter because for its understanding the reader should be familiar at least with the first three chapters of the book, where the basic notions, relations and invariants are introduced stepwise for the cases of local and global GALOIS coverings.

In the first chapter we define the category of abelian points. These are embedded cyclic surface singularities together with two weighted embedded curve germs intersecting at this point. Most important are finite morphisms and modifications in this category. We refer to part 1.1 - 1.18 of the contents in order to indicate the type of arithmetic games we have to play for finding the 0-local height contributions. Weighted (electronic) graphs of abelian points are introduced. They classify abelian points and store the contributions in a convenient manner. In the short chapter 2 we introduce orbital curves as surface germs along a weighted compact curve supporting abelian points. They form again a category with the important class of finite morphisms. Based on the results of the first chapter we are able to find two kinds of orbital heights: the EULER heights and the signature heights of orbital curves. The first generalizes the EULER number and the second the selfintersection index of a curve on a smooth surface. Essentially for the first deeper functorial understanding including a geometric Local-Global Principle for orbital curves is section 2.3. Using graphs of abelian points we define star-like weighted (atomic) graphs of orbital curves. The rational EULER and signature heights can be read off from them.

The functorial procedure is inductively extended from the smaller dimensions to orbital surfaces with natural arrangements using only natural weights and coefficients. These natural orbital surfaces are sufficient to understand the GALOIS part of functorial properties defining EULER and signature heights on this way. In 3.5 we introduce quotient points as surface germ around a quotient singularity together with three weighted curve germs through it. Cusp points are defined in a similar manner using four weighted curve germs instead of three intersecting at a cusp singularity. Both types of these new orbital points are classified by means of atomic graphs. Allowing also quotient and cusp points we introduce open orbital surfaces supported by the open complement of all cusp singularities of the underlying compact surface. The arithmetic functorial game with invariants extends to open orbital surfaces defining the two above heights for them. Graphs of orbital curves and points are connected to introduce weighted (molecular) graphs of orbital surfaces or, more precisely, of the corresponding arrangements. They store the 0- and 1-local contributions for the calculation of EULER and signature heights. Additionally, one needs the CHERN numbers of the supporting surfaces and of the normalized irreducible curves

of the arrangements. In the last section we present a formula containing the heights of the orbital curve components of arrangements as (weighted) summands. Altogether, we can say that the existence proof for the two above heights combines the arithmetic of singularities with RIEMANN-ROCH and, more generally, with equivariant K -theory in a new functorial manner using geometric Local-Global Principles.

In the next chapter we understand ball quotient surfaces $\mathbb{B}/\Gamma$, Γ a ball lattice, as open orbital surfaces denoted by $\mathbb{B}/\Gamma$. The support of the corresponding (open) arrangement $A(\Gamma)$ is the branch locus of the infinite GALOIS covering $\mathbb{B} \rightarrow \mathbb{B}/\Gamma$. The preimages of the 0- or 1-components are Γ -elliptic points or Γ -reflection discs, respectively. The weights come from ramification indices. They can be read off from the stationary subgroups of elliptic points, cusps or the discs, respectively. Up to a universal factor, the heights of each curve component of $A(\Gamma)$ turn out to be non-euclidean volumes of a fundamental domain in the covering disc with respect to its normalizing subgroup of Γ . Similarly, the heights of $\mathbb{B}/\Gamma$ appear, up to universal factors again, as volume of a Γ -fundamental domain with respect to the ball metric. On this way one gets via universal factors universal relations between EULER and signature heights of $\mathbb{B}/\Gamma$ and of the components of $A(\Gamma)$. This is an extension of HIRZEBRUCH-MUMFORD's proportionality theory in the case of the ball including all ball lattices now. The proportionality relations yield a strong criterion for (open) orbital surfaces to be ball quotients. We hope that it is precise. The proofs connect HIRZEBRUCH-MUMFORD's proportionality theory for neat lattices with a relative variant for subdiscs of the ball, due to the author, and the results of the previous chapters.

Because of the above identities of heights and volumes, the data of the arrangement $A(\Gamma)$ stored in its molecular graph allow to turn the CHERN-invariants of compact models of $\mathbb{B}/\Gamma$ into the volume of a Γ -fundamental domain and vice versa. The last three sections are dedicated to first applications. For given suitable point-curve configuration A on a fixed surface X we can endow the components with weights such that A becomes an arrangement $\mathbf{A}$ and X an orbital surface $\mathbf{X} = (X, \mathbf{A})$. There are infinitely many possibilities in general. In order to get an orbital ball quotient surface the above proportionality relations have to be satisfied. By means of the orbital decomposition of heights the relations are transformed to a system of diophantine equations for the unknown weights. We prove that this system can be solved effectively in any case. Moreover, it turns out that there are at most finitely many solutions. This is the effective Finiteness Theorem of 4.10. We apply it to some simple blowing up's of $\mathbb{P}^2$ and the preimage A of the complete quadrihedral consisting of six lines on $\mathbb{P}^2$ in a special position. All ball quotient arrangements supported by A can be found in 4.11. On

the one hand this example demonstrates the use of molecular graphs. On the other hand we know by some work of PICARD, TERADA, MOSTOW, DELIGNE that we get really ball quotient surfaces with monodromy lattices corresponding to certain classical systems of differential equations of PICARD-FUCHS-GAUSS-MANIN type. In the last section we interpret the Finiteness Theorem as existence criterion for (at most finitely many) ball metrics on a given open algebraic surface.

A further field of application is the theory of PICARD modular surfaces. Such surfaces are defined as (compactified) quotients $\mathbb{B}/\Gamma$ by certain arithmetic ball lattices Γ connected with imaginary quadratic number fields. They have rather complicated branch arrangements. The corresponding elliptic points and reflection discs together with stationary groups have been found by a hard number theoretic work of FEUSTEL during several years. The proofs would fill another monograph. So we present only the results and apply them to the rough KODAIRA classification of PICARD modular surfaces. Knowing the arrangements and heights we are able to calculate the CHERN numbers. The heights being essentially volumes of fundamental domains are calculated in the Appendix 5.A with number theoretic methods. It gives also an impression of the local height part of proofs in FEUSTEL's work. We start in the first sections demonstratively with the fine classification of the PICARD modular surfaces of EISENSTEIN and GAUSS numbers. In section 5.4 we prove that almost all PICARD modular surfaces are of general type with the fractional invariant c_1^2/c_2 coming arbitrary near to the value 3, which is extremely possible in surface theory. Examples of rational, elliptic and $K3$ surfaces are presented in section 5.5 using suitable sublattice of the full PICARD modular groups.

The historical order is the following: We started with 5.2, 5.3. Then 4.1 - 4.9 have been developed in close connection with chapter 1 and 5.3 - 5.5. Then the idea was born to generalize the multiplicative property of volumes of fundamental domains of lattices with respect to sublattices to orbital surface invariants with respect to GALOIS coverings. It leads to the introduction of our heights, arrangements, orbital surfaces. This general surface language, independently of discs and balls, enabled us to attack the theorems of 4.9 - 4.12. The next step was to extend the functorial ideas of heights to a more general class of orbital surfaces (with rational coefficients at arrangement components) with the first aim to present explicit HURWITZ formulas for CHERN invariants. This program is fulfilled in sections 6.1 - 6.9 for finite coverings of algebraic surfaces with (at most) cyclic singularities. In the last section 6.10 we present the following application with a glance to the well-known BOGOMOLOV-MIYAOKA-YAU Theorem: $c_1^2/c_2 \leq 3$ for smooth compact complex algebraic surfaces of general type. We prove that the sharper inequality $c_1^2/c_2 \leq 2$ holds for all surfaces allowing a finite

projection onto a ruled surface or onto $\mathbb{P}^2$ with a nice branch locus consisting of smooth transversally intersecting curves.

Suggestions. The author believes that the functorial ideas of constructing orbital heights can be inductively extended also to higher dimensions. In [HOV], chapter 6 we established precise dimension formulas for spaces of cusp forms of ball lattices Γ with a proof in the spirit of chapters 4 and 5. These dimensions can be read off also in a universal manner for each weight from the molecular graph of the arrangement $A(\Gamma)$ knowing additionally the volumes of fundamental domains. There are good reasons to assume that a functorial transfer of the dimension formulas in connection with logarithmic pluricanonical genera of open surfaces in the style of the chapters 1, 2, 3 and 6 is possible. It would produce an infinite series of heights and, maybe, explicit HURWITZ formulas for (logarithmic) pluricanonical genera with respect to finite surfaces coverings. At the end we ask for possibilities to construct orbital surfaces and heights for surfaces over fields of finite characteristic p and for arithmetical surfaces.