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# Symmetrization and Stabilization of Solutions of Nonlinear Elliptic Equations

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# Preface

This book mainly deals with a systematic study of a dynamical system approach to investigate the symmetrization and stabilization properties (as $|x|$ tends to infinity) of nonnegative solutions of nonlinear elliptic (both degenerate and nondegenerate) problems in asymptotically symmetric unbounded domains. To this end, we use a trajectory dynamical systems approach and the concept of its attractor. Recall that a dynamical system (DS) is a system which evolves with respect to time. To be more precise, a DS $(S(t), \Phi)$ is determined by a phase space $\Phi$ which consists of all possible values of the parameters describing the state of the system and an evolution map $S(t) : \Phi \to \Phi$ which allows us to find the state of the system at time $t > 0$ if the initial state at $t = 0$ is known. Very often, in biology, ecology, mechanics, and physics, more generally in the modeling of life science problems, the evolution of the system is governed by systems of differential equations. If the system is described by ordinary differential equations (ODEs)

$$u'(t) = F(t, u(t)) \tag{1}$$

for some nonlinear function $F : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$, we have a so-called finite-dimensional DS. In that case, the phase space $\Phi$ is some (invariant) subset of $\mathbb{R}^N$ and the evolution operator $S(t)$ is defined by

$$S(t)y_0 := y(t), \tag{2}$$

where $y(t)$ solves (1). We also recall that, in the case where Eq. (1) is autonomous (i.e., does not depend explicitly on time), the evolution operators $S(t)$ generate a semigroup on the phase space $\Phi$, i.e.

$$S(t_1 + t_2) = S(t_1)S(t_2), \qquad t_1, t_2 \in \mathbb{R}^+, \tag{3}$$

The qualitative study of DS in finite dimensions goes back to the beginning of the twentieth century with the pioneering works of Poincaré on the $N$-body problem and important contributions of Lyapunov on stability and of Birkhoff on minimal sets and the ergodic theorem. One of the most surprising and significant facts discovered at the very beginning of the theory is that even relatively simple equations can generate very complicated chaotic behaviors. Moreover, these types of systems are extremely sensitive to initial conditions (the trajectories with close but different initial data diverge exponentially). Thus, in spite of the deterministic nature of the system (we recall that it is generated by a system of ODEs, for which we usually have a unique solvability theorem), its temporal evolution is unpredictable on time scales larger than some critical time $T_0$ (which clearly depends on the error of approximation and on the rate of divergence of close trajectories) and can show typical stochastic behaviors. This fact was used by Lorenz to justify the so-called butterfly effect, a metaphor for the imprecision of weather forecasting. The theory of DS in finite dimensions has been extensively developed during the twentieth century, due to the efforts of many mathematicians (such as Anosov, Arnold, LaSalle, Sinai, Smale), and, nowadays, very much is known about the nontrivial dynamics of such systems, at least in low dimensions. In particular, it is known that very often, the trajectories of a chaotic system are localized, up to a transient process, in some subset of the phase space having a very complicated fractal geometric structure (e.g., locally homeomorphic to the Cartesian product of $\mathbb{R}^m$ and some Cantor set) which thus accumulates the nontrivial dynamics of the system (this is the so-called strange attractor). The chaotic dynamics on such sets are usually described by symbolic dynamics generated by Bernoulli shifts on the space of sequences. We also note that, nowadays, a mathematician has a large number of tools available for the extensive study of concrete chaotic DS in finite dimensions. In particular, we mention here different types of bifurcation theories (including the KAM theory and the homoclinic bifurcation theory with related Shilnikov chaos), the theory of hyperbolic sets, stochastic description of deterministic processes, Lyapunov exponents and entropy theory, dynamical analysis of time series, etc. In other words, in the mid-1970s of the twentieth century, we already had available highly developed finite-dimensional dynamical systems theory, and it has become vitally important to extend this theory to the evolution processes that are usually governed by partial differential equations (PDEs). In this case, the corresponding phase space $\Phi$ is some infinite-dimensional function space (e.g., $F := L^2(\Omega)$ or $F := L^8(\Omega)$ for some domain $\Omega \subset \mathbb{R}^N$, or Hölder space). Such DS are usually called infinite-dimensional. These classes of equations in the abstract setting can be written as

$$u'(t) = F(t, u(t)) \tag{4}$$

in an infinite-dimensional Banach space $\Phi$. We emphasize that for an infinite-dimensional DS generated by PDEs, the situation becomes much more complicated. Indeed, a first important difficulty which arises here is related to the fact that the analytic structure of a PDE is essentially more complicated than that of an ODE and, in particular, in general we do not have a unique solvability theorem as in the case of ODEs, so that even finding the proper phase space and the rigorous construction of the associated DS can be a highly nontrivial problem. In order to indicate the level of difficulties arising here, it suffices to recall that, for the three-dimensional Navier-Stokes system (which is one of the most important equations of mathematical physics), the required associated DS has not yet been constructed. Nevertheless, there exists a large number of equations for which the problem of the global existence and uniqueness of a solution has been solved. Thus, the question of extending the highly developed finite-dimensional DS theory to infinite dimensions arises naturally. Note that at present there are numerous monographs on infinite-dimensional dynamical systems generated by nondegenerate parabolic and hyperbolic equations (both autonomous and nonautonomous) in bounded domains and its application in various areas of mathematical physics (see, e.g., [1, 35, 67, 92, 98] and the references therein). In those books, the long-time dynamics of solutions is described in terms of a global attractor (uniform attractor in the nonautonomous case). It is important to note that the PDE equations in the books mentioned above possess regular structure (nondegenerate diffusion and nondegenerate chemotaxis) and belonging to the so-called dissipative partial differential equations (sometimes partially dissipative, when the underlying domain is unbounded or with hyperbolic equations in a bounded domain). We emphasize once more that the phase spaces $\Phi$ in those books mentioned above are appropriate infinite-dimensional function spaces. Nevertheless, it was observed in experiments that, up to a transient process, the trajectories of the DS considered are localized inside "very thin" invariant subsets of the phase space having a complicated geometric structure which, thus, accumulates all the nontrivial dynamics of the system. It was conjectured that these invariant sets are, in some proper sense, finite-dimensional and that the dynamics restricted to these sets can be effectively described by a finite number of parameters. Thus (when this conjecture is true), in spite of the infinite-dimensional initial phase space, the effective dynamics (reduced to this invariant set) is finite-dimensional and can be studied by using the algorithms and concepts of the classical finite-dimensional DS theory. Indeed, the above finite-dimensional reduction principle of dissipative (or partially-dissipative) PDEs in bounded domains has been given a solid mathematical grounding (based on the concept of the so-called global and exponential attractors) over the last four decades, starting from the pioneering papers of Ladyzhenskaya (see [77]) and continued successfully in [1, 35, 67, 92, 98] (see also references therein), mainly for evolution PDEs with more or less regular structure (e.g., uniformly parabolic, nondegenerate parabolic and nondegenerate hyperbolic) leading to finite dimensionality of their attractors. Here, as mentioned above, boundedness of the underlying domains plays a decisive role in obtaining finite fractal dimensional attractors of the associated semigroups. In [48] (see also references therein), we systematically studied infinite-dimensional dynamical systems and their attractors generated by a quite large class of nondegenerate parabolic type PDEs (both for second order and for fourth order) in unbounded domains (both in weighted and in uniformly local phase spaces) and proved that the fractal dimension of their attractors is infinite, so that the reduction principle fails in general. In spite of this new feature, that is, the infinite dimensionality of attractors, we managed to find asymptotics of their Kolmogorov's $\varepsilon$-entropy which has logarithmic type asymptotics.

It is worth mentioning that, in contrast to nondegenerate evolution equations, very little was known about the long-time dynamics of degenerate parabolic equations such as porous-medium equations, parabolic $p$-Laplacian, doubly nonlinear equations, and degenerate diffusion with chemotaxis and ODE-PDE coupling. In the books [46, 47] that were published very recently, we discussed these classes of degenerate parabolic equations in bounded domains in connections with the modeling of life science problems and observed some very interesting new features from the dynamical systems viewpoint, related to the attractors of such equations which cannot be observed in nondegenerate cases, namely:

(a) Infinite dimensionality of the attractor
(b) Polynomial asymptotics of its epsilon-Kolmogorov entropy
(c) Differences in the asymptotics of the epsilon-Kolmogorov entropy depending on the choice of the underlying phase spaces

Furthermore, for the evolution equations that are described by parabolic or hyperbolic type equations, the well-posedness, as already mentioned above, has been proved for a quite large class of nonlinearities. However, the nonlinearities involving the abovementioned PDEs that provides nonuniqueness of solutions usually possess some "pathological" nature (see, for instance, [52]) and in turn lead to multivalued semigroups from the dynamical systems viewpoint. I would like especially to emphasize that, in contrast to such equations where non-unicity appears for "pathological" class of nonlinearities, elliptic boundary value problems in unbounded domains, interpreted as evolution equations, naturally lead to nonuniqueness of solutions even for simple polynomial nonlinearities which as a consequence leads to multivalued semigroups, where the role of time (i.e., $t \geqslant 0$) is played by one of the unbounded directions of the underlying domain, say $t := x_1$. Indeed, it was a temptation to apply such a strong dynamical systems tool in order to study asymptotic of solutions of elliptic boundary value problem in unbounded domains. In other words, to consider state of equilibrium of bio-physico-chemical-mechanical systems from the point of view of dynamical systems. In order to apply DS approach to elliptic equations of the form given in *Chaps. 3–7* of this book, one has to take the following difficulties into account: firstly, the Cauchy initial value problem for such equations (prescribing both the value of solutions $u(0)$ and its normal derivative $u'(0)$ on the initial surface $t = 0$) is not well-posed. More precisely, in general, its unique solution either does not exist or blows up in a finite time. Secondly, the boundary value problem for elliptic equations may have a solution, but in general it is not unique. Indeed, Hadamard was the first to notice that the initial value problem for elliptic equations is ill-posed (see [66, Bk.I, Ch. II, §18]). In [8] this difficulty was overcome, where the attractor of an elliptic equation was constructed by means of the theory of semigroup of multivalued mappings. Unfortunately, the theory of semigroups of multivalued mappings does not provide theorems on the finite dimensionality of the attractors. Indeed such theorems are an important part of the theory of semigroups of single-valued mappings in infinite-dimensional spaces, which we called above a reduction principle. Since we are mainly interested in the finite fractal dimension of the attractors of semigroups associated with elliptic problems (so-called finite-dimensional reduction discussed above for parabolic/hyperbolic type equations), such multivalued semigroup approach is not suitable for our purposes. Therefore, the usage of infinite-dimensional dynamical systems methods mentioned above for elliptic problems in unbounded domains as well as finite-dimensional reduction of their dynamics requires new ideas and tools. Fortunately, these difficulties have been overcome in several interesting particular cases. We first mention the pioneering work by Kirchgassner [72] on small solutions of elliptic equations in infinite cylinders. His idea was to construct invariant manifolds, where the elliptic initial value problem is well-posed and a flow, or at least a semiflow, is defined (see also [62]). This idea was extended to large solutions, later, in the "parabolic," convection dominated limiting case of large wave speeds $\gamma \in \mathbb{R}$ (see [31] and [87]). Without such a restriction, the case of elliptic equations in a strip was treated (see [9] and [82]). Note that in [31] a similar problem to [8] was considered in the scalar case, when the elliptic equation contains a small parameter in front of second derivative, while the coefficient in front of first derivative is different from zero. In [8], the elliptic BVP was studied with Cauchy initial conditions $u(0) = u_0$ and $u'(0) = u_1$. In this case, the semigroup corresponding to this elliptic equation consists of a single-valued mappings, but the set on which it is defined is not described in an explicit form. We strongly believe that in the dynamical setting the most natural boundary value problem is to prescribe the initial value $u(0)$ at the $t = 0$ and impose the condition of boundedness of $u(t)$ as $t$ goes to infinity. We will consider throughout of this book this type of elliptic boundary value problem.

In this book, we overcome the abovementioned difficulties and some other ones (such as finite-dimensional reduction of dynamics) for an elliptic equation in unbounded domains by usage of the trajectory dynamical systems approach (the details are in *Chap. 3*) as well as symmetry and monotonicity properties of solutions of an elliptic equation (both nondegenerate and degenerate) in unbounded domains. In what follows, we use the following convention.

**Convention 1** The attractor of a trajectory dynamical system associated to an elliptic equation is called an elliptic attractor.

We specify below some of the topics covered by this book that we are mainly interested in:

1. To construct single-valued dynamical systems for nonlinear elliptic equations in unbounded domains
2. To find appropriate functional phase space (possibly not weighted space) in unbounded domains and determine a topology in the phase space guaranteeing existence of a global attractor
3. To find assumptions on nonlinearities under which all solutions converge as $t := x_1$ goes to infinity to the same limiting one-dimensional profile, irrespectively of the "initial" value $u_0$

4. To provide assumptions on the data, that is, the geometry of underlying domains, assumptions on nonlinearity, etc., for equations under consideration that guarantee finite-dimensional attractors
5. To find new effects imposed by the dynamical systems approach to nonlinear elliptic equations and to their elliptic attractors which was not observed in the parabolic or hyperbolic PDEs previously considered from the dynamical systems viewpoint
6. To understand whether asymptotic symmetry of the domain implies symmetry of elliptic attractors
7. To understand how smoothness or nonsmoothness of underlying domains is inherited by elliptic attractors
8. To provide assumptions guaranteeing that omega-limit sets of a solution of an elliptic equation is a singleton

This book consists of seven chapters.

*Chapter 1* consists of nine subsections. In *Sect. 1.1*, we introduce some functional spaces and study their properties that we will use in subsequent chapters. *Sections 1.2* and *1.3* are devoted to linear elliptic boundary value problems and the properties of Nemytskii operators in various functional spaces, respectively. *Sections 1.4* deals with the classical maximum principles as well as their versions for narrow and small domains, which are used in *Sects. 1.6*, *1.7*, and *1.8*. In *Sect. 1.5*, using maximum principles, we obtain explicit and uniform bounds that ensure the boundedness and the asymptotic dissipation of the solutions of semilinear elliptic equations in bounded (or unbounded) domains. These estimates are crucial to construct the attractive basin of trajectories in the dynamical system approach to elliptic equations. We will use these estimates in the following chapters. *Sections 1.6* and *1.7* are devoted to the sweeping principle, the moving plane method as well as sliding methods both in bounded and in unbounded domains and their role in the study symmetry and monotonicity properties of nonnegative solutions of nonlinear elliptic equations in unbounded domains. We use this symmetry and monotonicity properties of nonnegative solutions of nonlinear elliptic equations in unbounded domains for the complete characterization of the elliptic attractor. *Sections 1.8* and *1.9* are devoted to variational solutions and elliptic regularity for the Neumann problem for the Laplace operator on an infinite edge. We will use the results of these subsections to study the effects of nonsmoothness of the domain to nonsmoothness of elliptic attractor in *Chap. 2*. I would like especially to emphasize that, in some cases (of course, for the convenience of the reader), I present some well-known results both in this chapter as well as in the following chapters on the one hand referring to original sources and on the other hand giving proofs with preservation of the original notations of the authors which hopefully will make the book readable and self-consistent.

*Chapter 2* deals with the general theory of trajectory dynamical system and its attractor and its application for the study the asymptotics of solutions of semilinear elliptic boundary value problems in unbounded domains. *Chapter 2* consists of 14 subsections. In *Sects. 2.1* and *2.2*, we give definition of the fractal dimension and Kolmogorov's $\varepsilon$-entropy as well as their properties. Moreover, the definition of a global attractor for the semigroup as well as finite fractal dimensional reduction of its dynamics are also given in these sections. In *Sect. 2.3*, we give the complete characterization of asymptotic behavior of all bounded nonnegative solutions for semilinear elliptic boundary value problems in a two-dimensional rectangle. *Sections 2.4–2.12* are devoted to the trajectory dynamical systems approach to semilinear elliptic system in general unbounded domains, the existence of at least one solution, the regularity of solutions, basic definitions of trajectory attractors, and its regularity as well as to some examples of trajectory attractors and its generalizations. In *Sect. 2.13*, we study how nonsmoothness of underlying unbounded domain transfers to nonsmoothness of trajectory attractors. *Section 2.14* is devoted to the relation of nonautonomous parabolic equations in cylindrical domain with its elliptic counterpart via a traveling wave ansatz, and we study the latter from the dynamical systems viewpoint.

In *Chaps. 3* and *4*, we consider semilinear elliptic boundary value problems in the quarter-space and study asymptotic behavior of its nonnegative solutions when the underlying dimension of the quarter-space is less than or equal to three and four, respectively. Here the dimension of the underlying domain plays a very essential role, because the technique for each dimension providing asymptotic behavior of solutions in terms of the trajectory attractor depends heavily on the dimension of the quarter-space. Indeed we show that under very natural dissipativity assumptions, as well as sign condition on the nonlinearity (any polynomial nonlinearity satisfies this dissipativity condition), the trajectory attractor (that by definition captures all the asymptotic behavior of nonnegative solutions in the quarter-space) consists of all nonnegative solutions of the same semilinear elliptic equation, however, in the half-space. Using the symmetry and monotonicity results for nonnegative solutions in such domains (the technique in this chapter for the proving of symmetry and monotonicity results requires dimension restrictions and is based on the moving plane method considered in *Sect. 1.7*), we can describe the asymptotic behavior of solution and prove that the elliptic attractor is one-dimensional and any solution of the elliptic BVP in the quarter-space converges to a unique solution of a second-order ODE, indicating remarkable one-dimensional reduction of dynamics on the attractor. We emphasize that indeed such trajectory DS approach provides an elegant proof of a stabilization and symmetrization results that in general are really difficult to obtain using pure elliptic techniques, due to the unboundedness of the underlying domains in any directions and the possible appearance of oscillations during the limiting procedure. These chapters consists of four subsections, which deals with formulation of the problem (*Sects. 3.1* and *4.1*), a priori estimates and solvability results (*Sects. 3.2* and *4.2*), the existence of attractor for associated trajectory dynamical systems (*Sects. 3.3* and *4.3*), and symmetry and stabilization results (*Sects. 3.4* and *4.4*), respectively.

In *Chap. 5*, we consider the same elliptic boundary value problem as was studied in *Chaps. 3* and *4*. However, in this chapter, the dimension of the underlying domain is less than or equal to five. We aim at extending the results from dimensions three and four to the dimension five. Unfortunately we cannot apply the techniques from the previous chapters. We show how to avoid technical difficulties arising in this case of dimension five and obtain new type of asymptotics in contrast to previous chapters. Indeed, under the assumptions that all equilibria are stable, we prove that:

(a) ANY nonnegative bounded solution of a semilinear elliptic BVP in the quarter-space converges to a uniquely defined solution of a second-order ODE.
(b) ANY nonnegative bounded solution in the half-space converges to a uniquely defined constant solution.

In *Chap. 6*, we present completely new Liouville-type theorem for two classes of nonlinearities which give us the possibility to consider the same semilinear elliptic boundary value problem in any dimension. It is worth noting that these new Liouville-type results for any dimension are of independent interest for the solutions in half-spaces with homogeneous Dirichlet boundary conditions or in the whole space. In contrast to the results of the *Chap. 5*, one of the main points is that the results of *Chap. 6* hold in any dimensions without any assumption on a solution other than its boundedness. In *Chap. 5*, the main results were obtained in lower dimensions of the underlying domains and under the additional assumption that the solutions are stable, which among others requires $C^1$-differentiability of the nonlinearities. However, the assumptions on the nonlinearity $f(s)$ imposed in the *Chap. 5* are incompatible with the assumptions for the nonlinearity in *Chap. 6* which are crucial to prove new Liouville-type results. Using these new Liouville-type results, we completely characterize the limiting behavior of solutions of semilinear elliptic boundary value in the quarter-space as well as in the half-space based on the trajectory dynamical systems approach. It is worth noting that all of these results required completely new ideas, tools, and techniques.

Moreover in this *Chap. 6*, in contrast to the previous *Chaps. 3–5*, we compare two different approaches, namely, PDEs and dynamical systems approach discussing their individual advantages, as regards describing asymptotic profiles of solutions both in the quarter- and half-spaces in any dimension. To prove there exists a one-dimensional elliptic attractor, we use, among others, the sliding method which was discussed in *Sect. 1.7*. *Chapter 6* consists of four subsections. *Section 6.1* is devoted to the formulation of the problem as well as the introduction of two classes of nonlinearities: sign-changing and sign-preserving. In *Sect. 6.2*, we deal with the PDE approach to an elliptic boundary value problem both in the quarter- and half-spaces and study some properties of solutions based on, among others, sliding methods that are relevant for determining the asymptotic of solutions. *Section 6.3* deals with new type of Liouville results for bounded nonnegative solutions which on the one hand is of independent interest (it holds in any dimension without any assumption on a solution other than its boundedness) and on the other hand plays a decisive role in the study of uniqueness and one-dimensional symmetry of the limiting profiles of solutions as $x_1$ goes to infinity. In *Sect. 6.4*, we apply the trajectory DS approach to study symmetrization and stabilization properties (as $x_1$ goes to infinity) of nonnegative solutions for the equations in *Sect. 6.1* and compare this DS approach with the PDE approach developed in Sect. *6.2*.

*Chapter 7* deals with the asymptotic behavior of solutions of quasilinear elliptic problems over the quarter-space, and with similar problems over the half-space, and consists of four subsections. In *Sect. 7.1*, we formulate the same type of elliptic boundary value problem for the general $p$-Laplacian as was done in *Chap. 6* for the standard Laplacian and discuss the difficulties that arise in this case. We introduce two classes of nonlinearities: sign-changing and sign-preserving nonlinearities satisfying new assumptions that take into account the $p$-Laplacian nature of the equations. Since there is no strong comparison principle in general for quasilinear elliptic problem, we have to adopt rather different approaches in many key steps previously done in *Chaps. 3–6*. In *Sect. 7.2*, we present some basic results which will be needed in our investigation of the half- and quarter-space problems for $p$-Laplacian. A crucial ingredient here is a simple weak sweeping principle which is a consequence of the weak comparison principle for the $p$-Laplacian. We show that in many situations, it is possible to use the weak sweeping principle to replace the moving plane or sliding method which are based on the strong comparison principle for the Laplacian case which were frequently used in *Chaps. 3–6*. Using the weak sweeping principles, we prove new Liouville-type results for $p$-Laplacian equation in the unbounded domains mentioned above. *Section 7.3* deals with the asymptotic behavior of solutions of $p$-Laplacian equations using these new Liouville-type results for $p$-Laplacian equations. *Section 7.4* is devoted to corresponding results in the case of $p$-Laplacian with those of the semilinear case, existence of solution, as well as exact multiplicity results.

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