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Renato Portugal

# Quantum Walks and Search Algorithms

Second Edition

 Springer

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*To my father (in memoriam)*

# Preface

This is a textbook about *quantum walks* and *quantum search algorithms*. The readers will take advantage of the pedagogical aspects and learn the topics faster and make less effort than reading the original research papers, often too convoluted. The exercises and references allow the readers to deepen their knowledge on specific issues. Guidelines to use or to develop computer programs for simulating the evolution of quantum walks are also available.

Almost nothing can be extracted from this book if the reader is unfamiliar with the postulates of quantum mechanics, described in the second chapter, and the material on linear algebra described in Appendix A. Some extra bases are required: It is desirable that the reader has (1) notions of quantum computing, including the circuit model, references of which are provided at the end of Appendix A, (2) notions of graph theory, references of which are provided at the end of Appendix B, and (3) notions of classical algorithms and computational complexity. Any undergraduate or graduate student with this background can read this book. Some topics addressed in this second edition are currently active research areas with impact on the development of new quantum algorithms. Because of that, researchers working with quantum computing may find this book useful.

The second edition brings at least three main novelties: (1) a new chapter on the staggered quantum walk model—Chap. 8, (2) a new chapter on the element distinctness problem—Chap. 10, and (3) a new appendix on graph theory—Appendix B. Besides, the chapter on quantum-walk-based search algorithm—Chap. 9—was rewritten, the presentation has been simplified, and new material has been included.

Corrections, suggestions, and comments are welcome, which can be sent through Web page ([qubit.lncc.br](http://qubit.lncc.br)) or directly to the author by email ([portugal@lncc.br](mailto:portugal@lncc.br)).

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In January and February 2018, I gave a short course on quantum-walk-based search algorithms at the Tohoku University under the invitation of Dr. Etsuo Segawa. I thank the students and researchers that attended the course, who raised interesting discussion topics, helping to improve some chapters of the new edition of this book.

I thank Tom Spicer and Cindy Zitter from Springer for encouraging me to write the second edition, which turned out to be an opportunity for fixing many problems of the first edition and improving the book by adding new material. I hope to have introduced fewer problems this time.

I thank the support of the National Laboratory of Scientific Computing (LNCC), the funding agencies CNPq, CAPES, and FAPERJ, and the scientific societies SBMAC and SBC.

Last but not least, from the bottom of my heart, I thank my family, wife and sons, for giving support and amplifying my inner motivation.

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