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Free Boundary Problems

Regularity Properties Near the Fixed Boundary

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Preface

Free boundary problems (FBPs) belong to the most striking component of the modern theory of partial differential equations (PDEs). The expression FBP refers to a problem in which one or several variables are governed in different subdomains of the space, or space-time, by the different state laws. These subdomains are a priori unknown and have to be determined as a part of the problem. The boundaries of these unknown subdomains are called the *free boundaries*.

FBPs are the typical example of nonlinear problems where singularities arise. Therefore, a particular direction in FBPs has been to study the regularity properties of solutions and those of the free boundaries. Such questions are important for experiments and numerics and are usually considered extremely hard. Since the free boundary is not known a priori, the classical techniques of elliptic/parabolic PDEs do not apply. In the last two decades, many new approaches, combining the ideas from PDEs with ones from geometric measure theory, calculus of variations, harmonic analysis, and so on, have been developed and provided interesting results. This book treats some parts of this subject and its recent development.

To be precise, this book is devoted to the so-called *obstacle-type problems*, a class of FBPs which may be characterized by the following property: gradient of a solution is continuous across the interface. For several elliptic and parabolic one- and two-phase obstacle-type problems, the qualitative properties of solutions and free boundaries near the fixed boundary of a domain are studied. It is supposed that the Dirichlet data are prescribed on the fixed boundary.

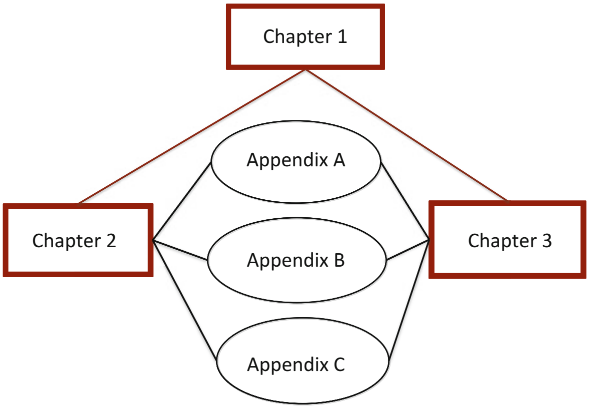
The book is divided in to three chapters and three appendices. Figure 1 illustrates the main dependencies among the book's components.

Chapter 1 is introductory in nature. It contains a narrative introduction into the field of FBPs as well as a brief observation of other parts of the book and the outlines of the main steps.

Chapters 2 and 3 form the core of the book. They can be treated independently. The largest part of the text contains results on parabolic problems. It includes the whole Chap. 2 and a half of Chap. 3.

In Chap. 2, the complete study of regularity issues up to the fixed boundary is carried out for the case of the no-sign parabolic obstacle-type problem with the

Fig. 1 The main dependencies among chapters



homogeneous Dirichlet boundary data. As the first step the optimal regularity of solutions is established; after that the analysis of global solutions is given. Based on these results, the fine properties of the free boundary are obtained, which boil down to proving a parabolic-tangential touch between the free and fixed boundaries. The latter in turn can be used to show C^1 properties of the free boundary.

In the case of the nonhomogeneous Dirichlet boundary data, the situation is much more complicated, especially for the two-phase problems. Even to prove the optimal regularity of solutions is not easy; it requires a special control for dependence of all the estimates on the distance to the fixed boundary. These questions are discussed in Chap. 3. Some of the results obtained there are stronger than ones known for the classical obstacle problem.

In Appendix A, one can find all necessary information about various monotonicity formulas which are the most important technical tools in studying the free boundary problems. In Appendix B, we recall and explain several general facts. Most of these auxiliary results are known, but probably not well known in the context used in this book. For the reader's convenience, we collect in Appendix C some facts concerning various problems with free boundaries. These facts are involved essentially in our arguments. All these appendices are very much in regular use in Chaps. 2 and 3.

The intended audience of this book includes graduate students and young researchers entering this field of mathematics. The reader is assumed to be familiar with classical calculus and the standard elliptic/parabolic theory (maximum/comparison principle, interior/boundary estimates, compactness arguments, etc.).

This text is based on the revised version of my habilitation thesis at Saarland University. Parts of the book have been used as material for graduate courses on FBPs that I taught at Saarland University (2010, 2011, and 2017) and at Peoples' Friendship University of Russia in Moscow (2016).

I would like to mention just a few names standing for the long list of persons who contributed to this book in one way or the other. First and foremost, I am deeply indebpted to Nina Uraltseva. She led me to the Diploma and PhD. We have been collaborating for more than 20 years. I am very grateful to her for support, numerous discussions and valuable advice. My special thanks go to Martin Fuchs. Without his kind support and sometimes also his pressure, this book would never have been completed. Furthermore, I wish to thank Michael Bildhauer who carefully read the manuscript and gave me useful suggestions. I would like to express my sincere thanks to my coauthors Henrik Shahgholian and Norayr Matevosyan for the contributions they have made to our joint publications. Finally, I am very thankful to my family for their efforts to not disturb me too much and sharing with me the pressure associated with a project like this.

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Saarbrücken, Germany — Darya Apushkinskaya
July 2018

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Basic Notation and Conventions

$z = (x, t)$	Generic points in $\mathbb{R}^{n+1}$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^1$
$x = (x_1, x') = (x_1, x_2, \dots, x_n)$	Points in $\mathbb{R}^n$, (if $n \geq 2$)
$\lvert x \rvert$	Euclidean norm of x in $\mathbb{R}^n$
$x \cdot y$	Inner product in $\mathbb{R}^n$
$e_1, \dots, e_n$	Standard basis in the x-space $\mathbb{R}^n_x$
e_0	Standard basis vector in the t-space $\mathbb{R}^1_t$
$\mathbb{R}^n_+$	$\{x \in \mathbb{R}^n : x_1 > 0\}$
$\mathbb{R}^{n+1}_+$	$\{(x, t) \in \mathbb{R}^{n+1} : x_1 > 0\}$
$\mathbb{R}^{n+1}_-$	$\{(x, t) \in \mathbb{R}^{n+1} : x_1 < 0\}$
Π	$\{(x, t) \in \mathbb{R}^{n+1} : x_1 = 0\}$
Π_r	$\{(x, t) \in \Pi : \lvert x \rvert \leqslant r,\ -r^2 < t \leqslant 0\}$
$\Pi_r(t^0)$	$\Pi_r \cap \{t = t^0\}$
$B_r(x)$	Open ball in $\mathbb{R}^n$ with center x and radius r
$B^+_r(x)$	$B_r(x) \cap \mathbb{R}^n_+$
B_r, B^+_r	$B_r(0)$, $B^+_r(0)$
$S_r(x^0)$	$\{x \in \mathbb{R}^n : \lvert x - x^0 \rvert = r\}$
S_r	$S_r(0)$
$Q_r(z^0) = Q_r(x^0, t^0)$	$B_r(x^0) \times]t^0 - r^2, t^0]$
$Q^+_r(z^0)$	$Q_r(z^0) \cap \mathbb{R}^{n+1}_+$
Q_r, Q^+_r	$Q_r(0, 0)$, $Q^+_r(0, 0)$
$\mathscr{Q}_r$	$B_r \times]0, 1]$
$\partial' Q_r(z^0)$, $\partial' \mathscr{Q}_r$	Parabolic boundary, i.e., the topological boundary minus the top of the cylinder
$\mathscr{Z}_{r,s}(z^0)$	$B_r(x^0) \times]t^0 - s, t^0 + s[$
$\mathscr{Z}^+_{r,s}(z^0)$	$\mathscr{Z}_{r,s}(z^0) \cap \mathbb{R}^{n+1}_+$
$K_r(z^0)$	$\{\lvert x_1 - x^0_1 \rvert < r\} \times \{\lvert x' - (x^0)' \rvert < r\} \times]t^0 - r^2, t^0[$

$K_r^+(z^0)$	$K_r(z^0) \cap \mathbb{R}_+^{n+1}$
$\mathbb{K}_\delta(x^0)$	$\left\{x \in \mathbb{R}^n : x_1 - x_1^0 > \lvert x' - \left(x^0\right)' \rvert \tan(\delta)\right\}$
$\mathbb{K}_\delta$	$\mathbb{K}_\delta(0)$
∂_t	Differential operator with respect to t
D_i	Differential operator with respect to x_i
$D' = (D_2, \ldots, D_n)$	Tangential gradient
$D = (D_1, D')$	Spatial gradient
$\nabla u = (Du, \partial_t u)$	The complete gradient of u in the space $\mathbb{R}_x^n \times \mathbb{R}_t^1$
$D^2 u$	$D(Du)$
$D^3 u$	$D(D^2 u)$
$H = \Delta - \partial_t$	The heat operator
ν, e	Arbitrary unit vectors
$e \perp \nu$	e is orthogonal to ν
D_ν	Operator of differentiation along the direction ν
$D_{\nu\nu}$	$D_\nu\,(D_\nu)$
$\mathscr{H}^m$	m-dimensional Hausdorff measure
$\mathscr{L}^m$	m-dimensional Lebesgue measure
v^+, v^-	$\max\{v, 0\}$, $\max\{-v, 0\}$.

With a slight abuse of notation, we will consider the sets $E \subset \mathbb{R}_x^n$ as well as $E \subset \mathbb{R}^{n+1} = \mathbb{R}_x^n \times \mathbb{R}_t^1$. In both cases, χ_E denotes the characteristic function of the set E, while ∂E stands for the topological boundary of E. We emphasize that the precise meaning of E will always be evident by the context.

$\fint_E \ldots$ stands for the average integral over the set E, i.e.,

$$\fint_E \cdots = \frac{1}{\text{meas}\,\{E\}} \int_E \ldots;$$

The reader is assumed to be familiar with the classical

- Hölder spaces $C^{k,\alpha}(E)$
- Lebesgue spaces $L^p(E)$ with the norm $\|\cdot\|_{p,E}$, $1 < p \leqslant \infty$

as well as with the notions of the

- Isotropic and anisotropic Sobolev spaces $W^{1,p}(E)$, $W^{2,p}(E)$, and $W_p^{2,1}(E)$ with the norms

$$\|u\|_{W^{1,p}(E)} = \|Du\|_{p,E} + \|u\|_{p,E},$$
$$\|u\|_{W^{2,p}(E)} = \|D(Du)\|_{p,E} + \|u\|_{p,E},$$
$$\|u\|_{W_p^{2,1}(E)} = \|\partial_t u\|_{p,E} + \|D(Du)\|_{p,E} + \|u\|_{p,E},$$

respectively.

Here, for Hölder, Lebesgue, and isotropic Sobolev spaces, we follow the definitions and notations as introduced in book [AF03]. In particular, local variants are denoted by $L^p_{loc}(E)$, $C_{loc}(E)$, $W^{1,2}_{loc}(E)$, etc. For definitions of anisotropic Sobolev spaces, we refer the reader to [Kry08].

By $\zeta = \zeta(|x|)$ is denoted a time-independent cut-off function belonging to $C^2(B_{1/2})$, having support in $B_{1/4}$, and satisfying $\zeta \equiv 1$ in $B_{1/8}$.

$\xi = \xi(|x|)$ stands for a time-independent cut-off function belonging to $C^2(B_2)$, having support in B_2, and satisfying $\xi \equiv 1$ in B_1.

$\xi_{r,x^0}(x) = \xi\left(\dfrac{|x - x^0|}{r}\right)$.

It remains to mention four conventions which are not restated each time:

- The indices i, j always vary from 1 to n, while the indices τ, μ vary from 2 to n. Repeated indices indicate summation, for example, $a^{ij}x_ix_j = \sum\limits_{i,j=1}^{n} a^{ij}x_ix_j$.
- If necessary (and possible), we usually pass to subsequence without relabeling.
- We use letters M, N, A, L, and C (with or without indices) to denote various constants. To indicate that, say, N depends on some parameters, we list them in parenthesis: $N(\dots)$.
- Positive constants are not relabeled. Moreover, they are not necessarily being the same in any two occurrences.