

Nečas Center Series

Editor-in-Chief:

Josef Málek, Charles University, Czech Republic

Managing Editor:

Beata Kubiš, Czech Academy of Sciences, Czech Republic

Editorial Board:

Peter Bastian, Universität Heidelberg, Germany

Miroslav Bulíček, Charles University, Czech Republic

Andrea Cianchi, Università degli Studi di Firenze, Italy

Camillo De Lellis, Universität Zürich, Switzerland

Eduard Feireisl, Czech Academy of Sciences, Czech Republic

Volker Mehrmann, Technische Universität Berlin, Germany

Luboš Pick, Charles University, Czech Republic

Milan Pokorný, Charles University, Czech Republic

Vít Průša, Charles University, Czech Republic

KR Rajagopal, Texas A&M University, USA

Christophe Sotin, California Institute of Technology, USA

Zdeněk Strakoš, Charles University, Czech Republic

Endre Süli, University of Oxford, UK

Vladimír Šverák, University of Minnesota, USA

Jan Vybíral, Czech Technical University, Czech Republic

The Nečas Center Series aims to publish high-quality monographs, textbooks, lecture notes, habilitation and Ph.D. theses in the field of mathematics and related areas in the natural and social sciences and engineering. There is no restriction regarding the topic, although we expect that the main fields will include continuum thermodynamics, solid and fluid mechanics, mixture theory, partial differential equations, numerical mathematics, matrix computations, scientific computing and applications. Emphasis will be placed on viewpoints that bridge disciplines and on connections between apparently different fields. Potential contributors to the series are encouraged to contact the editor-in-chief and the manager of the series.

More information about this series at <http://www.springer.com/series/16005>

Miroslav Bulíček • Eduard Feireisl • Milan Pokorný
Editors

New Trends and Results in Mathematical Description of Fluid Flows

 Birkhäuser

Editors

Miroslav Bulíček
Faculty of Mathematics and Physics
Charles University
Prague, Czech Republic

Eduard Feireisl
Institute of Mathematics
Czech Academy of Sciences
Prague, Czech Republic

Milan Pokorný
Faculty of Mathematics and Physics
Charles University
Prague, Czech Republic

ISSN 2523-3343

ISSN 2523-3351 (electronic)

Nečas Center Series

ISBN 978-3-319-94342-8

ISBN 978-3-319-94343-5 (eBook)

<https://doi.org/10.1007/978-3-319-94343-5>

Library of Congress Control Number: 2018953362

Mathematics Subject Classification (2010): 35D99, 35L02, 35Q30, 35Q31, 76B03, 76D03, 76M10, 76N10, 76T10

© Springer Nature Switzerland AG 2018

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Switzerland AG.

The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Contents

Preface	ix
----------------------	----

Dominic Breit

An Introduction to Stochastic Navier–Stokes Equations

1 Introduction	1
2 Preliminaries	4
2.1 Stochastic processes	4
2.2 Stochastic integration	7
2.3 Itô’s Lemma	9
2.4 Stochastic ODEs	10
2.5 Stochastic analysis in infinite dimensions	17
2.6 Tools for compactness	20
3 Incompressible fluids	21
3.1 The approximated system	23
3.2 Compactness	27
3.3 The system on the new probability space	30
4 Compressible fluids	32
4.1 A priori estimates	36
4.2 Compactness	40
4.3 Strong convergence of the density	45
Acknowledgment	49
References	50

Yann Brenier

Some Concepts of Generalized and Approximate Solutions in Ideal Incompressible Fluid Mechanics Related to the Least Action Principle

1 The Least Action Principle for an ideal incompressible fluid	53
--	----

1.1	The configuration space of an incompressible fluid	53
1.2	The Euler equations	54
1.3	Geometric interpretation of the Euler equations	54
1.4	The Least Action Problem (LAP)	56
2	From the Least Action Problem to the polar decomposition of maps	57
2.1	The semi-discrete Least Action Problem	57
2.2	The mid-point problem and the polar decomposition of maps	58
3	Generalized solutions to the Least Action Problem	61
3.1	The concept of generalized flows	61
3.2	The weak formulation of the LAP	62
3.3	The semi-discrete version of the weak LAP	63
4	Results on the generalized Least Action Problem	63
4.1	Continuity of generalized solutions with respect to data	64
4.2	Example of generalized solutions	65
4.3	Two phase flows in one space dimension	66
5	A dissipative least action principle for approximations of the Euler equations	67
5.1	Finite-dimensional examples	67
5.2	The main example and the Vlasov–Monge–Ampère system	68
5.3	Conservative solutions à la Bouchut–Ambrosio	69
5.4	Rewriting of the action for “good” curves	69
5.5	Gradient-flow solutions as special least-action solutions	70
5.6	Global dissipative solutions of the gradient-flow	70
5.7	A proposal for a modified action	71
6	Stochastic and quantum origin of the dissipative least action principle	71
6.1	Localization of a Brownian point cloud	71
6.2	An alternative viewpoint: the pilot wave	72
6.3	Large deviations of the pilot system	73
	References	73

Didier Bresch and Pierre-Emmanuel Jabin

Quantitative Regularity Estimates for Compressible Transport Equations

	Preface	77
1	Lagrangian approaches	79
1.1	The Cauchy–Lipschitz theory	79

1.2 Lagrangian estimates for $u \in L^1(0, T; W^{1,p}(\Pi^d))$	80
1.3 An Eulerian formulation	85
2 Examples of Eulerian approaches: Renormalized solutions	90
2.1 Basic notions of renormalized solutions	91
2.2 Proving the renormalization property: Commutator estimates	94
2.3 The log log scale for compressible transport equations	97
2.3.1 Technical preliminaries	97
2.3.2 Propagating regularity with weights	99
2.3.3 The final estimate	102
3 Example of application: A coupled Stokes system	104
3.1 The compressible Navier–Stokes system	105
3.2 The result on the Stokes system	106
3.3 Sketch of the proof of Theorem 3.2	106
3.3.1 Construction of approximate solutions	107
3.3.2 Energy estimates	107
3.3.3 Stability of weak sequences: Compactness	108
References	110

Christian Rohde

**Fully Resolved Compressible Two-Phase Flow:
Modelling, Analytical and Numerical Issues**

Preface	115
1 The sharp interface approach	118
1.1 The Euler equations for one- and two-phase flow	118
1.1.1 Thermodynamical framework	118
1.1.2 Isothermal flow	121
1.1.3 Sharp interface solutions	124
1.2 The Riemann problem	130
1.2.1 The rotated Riemann problem and Lagrangian setting	130
1.2.2 Elementary waves and phase boundaries	132
1.2.3 The Riemann problem for one-phase flow	135
1.2.4 The Riemann problem for two-phase flow	137
1.3 A finite volume moving mesh method	143
1.3.1 Finite volume schemes on moving meshes	144
1.3.2 Finite volume schemes on moving meshes with interface tracking	148
1.3.3 Thermodynamical consistency in one spatial dimension	151
1.3.4 Numerical results for a single bubble	154
2 The diffuse interface approach	155

2.1	The Navier–Stokes equations for one-phase flow	156
2.1.1	Modelling and thermodynamical consistency	156
2.1.2	A thermodynamically consistent finite-volume scheme	158
2.2	The classical Navier–Stokes–Korteweg equations for two-phase flow	160
2.2.1	Modelling and thermodynamical consistency	160
2.2.2	A numerical illustration	163
2.3	Relaxed Navier–Stokes–Korteweg equations for two-phase flow	164
2.3.1	Modelling and thermodynamical consistency	164
2.3.2	A thermodynamically consistent finite-volume scheme	169
2.4	Well-balanced Navier–Stokes–Korteweg equations for two-phase flow	172
	Acknowledgement	176
	References	176

Preface

This special volume consists of four surveys that are focussed on several aspects in fluid dynamics. The basis for these surveys were series of lectures delivered by Dominic Breit (Heriot-Watt University Edinburgh, United Kingdom), Yann Brenier (Ecole Polytechnique, Palaiseau, France), Pierre-Emmanuel Jabin (University of Maryland, USA) and Christian Rohde (Universität Stuttgart, Germany) at the EMS School in Applied Mathematics *Mathematical Aspects of Fluid Flows* held at Kácov, Czech Republic, May 28–June 2, 2017.

The four surveys cover the following subjects: stochastic partial differential equations in fluid mechanics, different concepts of solutions in ideal fluid mechanics connected with the least action principle, new views on the theory of the transport and continuity equations in connection with the compressible Navier–Stokes equations and related models, and modelling, mathematical and numerical analysis of models as well as the numerical solution of equations describing multi-phase fluid flows.

The main objective of the series of the Kácov Schools is to present new, modern methods, tools and results in the mathematical theory of compressible and incompressible fluids, and more complex systems of partial differential equations. Such a goal must be, however, motivated by studying physically relevant problems. In particular, the rigorous analytical results should lead to good understanding of the behaviour of the model, should predict its limitations and should indicate the kind of numerical method that could be used in order to solve the problem in a stable, accurate and efficient manner. Hand-in-hand with these mathematical properties, the analysis of the models should also lead to the relevant qualitative predictions of the model that are compatible with the physical expectations/experiments. Furthermore the model must obey basic physical principles, such as energy conservation or an entropy inequality. We believe that each of the four surveys fulfills these requirements and presents the most recent results and points of view, and addresses important problems in the mathematical theory of fluids.

One of the popular recent directions is to include suitable small perturbations into the models, which may be of numerical or empirical nature, or may originate in physical uncertainties (such as thermodynamic fluctuations occurring in fluid flows). This issue is discussed in the first part of the volume authored by Dominic Breit. He explains how the classical (deterministic) equations describing fluid flow can be modified to include stochastic effects, leading to stochastic partial

differential equations, where all of the quantities involved (such as the density, the velocity and the temperature) are defined on a probability space, and he shows how to include an appropriate Wiener process into the equations. In the introduction the author presents a nice overview of the methods used in the theory of stochastic PDEs. In the rest of the survey, the author discusses two notions of solution: a deterministic equation with random coefficients (semi-deterministic approach) and a fully stochastic problem (finite energy weak martingale solutions), and he highlights the strong and the weak points of both approaches. The available theory for both incompressible and compressible fluids is described in detail. Finally, the main steps of the existence proofs are presented.

Yann Brenier discusses various concepts of generalized and approximate solutions in the mathematical theory of ideal (inviscid) incompressible fluids. The central concept is the relation between solutions of the fluid system in the Eulerian description given by the standard Euler system of partial differential equations and a variational approach based on the least action principle. The least action principle characterizes the admissible trajectories as those minimizing the action functional. It is shown that smooth solutions of the Euler system when restricted to a sufficiently short time interval comply with the least action principle, while they represent critical points of the latter when the time lap is large. Furthermore, various concepts of generalized solutions to the least action principle are introduced and solvability of the latter in this new framework is discussed. Finally, the approximation of the Euler equations, based on a modified least action principle taking into account the energy dissipation, is considered. The intimate relationship of this concept to stochastic and quantum phenomena is discussed in the final part of the chapter.

The next part of the volume is the survey written by Didier Bresch and Pierre-Emmanuel Jabin and is focussed on the regularity and qualitative theory and estimates for the advection equation driven by a nonsmooth velocity field. The authors present a nice overview of the available results, but also show the essential tools and methods used in the most recent results. These are then used to obtain new results for complex systems where the transport equation is coupled to other PDE's, e.g., to the compressible Navier–Stokes system. The aim is also to impose relevant and available estimates for the velocity field or for the unknown, i.e., one cannot impose a bound on the divergence of the velocity or upper or lower bounds on the unknown itself. The whole theory is therefore built for situations when the velocity belongs to a certain Bochner–Sobolev space, which is the natural candidate appearing in the theory of compressible fluids. The survey is split into three main parts: the first one deals with the Lagrangian approach and the logarithmic scale for advective equations; the second part concerns the Eulerian description of the problem and takes into account the power of the technique of renormalized solutions to deduce compactness results for the compressible Navier–Stokes system; the last part provides a beautiful example of how the method introduced by the authors can be applied to the Stokes system coupled with a nonmonotone pressure law.

The last survey, authored by Christian Rohde, ranges from mathematical modelling to sophisticated numerical methods for multi-phase compressible flows. He considers the compressible free flow of homogeneous fluids that occur in liquid and vapour phases and he aims to describe also phase-change phenomena. Two methods in particular are discussed in the survey. First, models which display the phase boundary as a sharp interface are discussed, and second, diffuse interphase models are considered. For both classes of models the associated thermodynamic framework is set up, the applicability of the model is discussed, and finally a suitable numerical scheme is introduced. The most developed among these is the isothermal sharp interphase model, which is understood as a free boundary value problem with appropriate coupling conditions across the interface. These conditions are of the form of an algebraic constitutive relation and the choices of kinetic relations, leading to consistency, well-posedness and a thermodynamic theory, are discussed. Ultimately, these results appear then to be only a weak basis for (multi-dimensional) numerics but it turns out that an appropriate Riemann solver is a key for the construction of moving-mesh finite volume methods for the thermodynamically consistent tracking of interfaces with mass transfer. The sharp interphase model is not able to describe topological changes of the interface. This issue is then solved by using the Navier–Stokes–Korteweg model, where the free energy functional is extended by higher-order terms. The stress tensor is then changed accordingly in order to retain thermodynamical consistency. Finally, a numerical discretization that is able to deal with higher-order and nonlocal terms in the stress tensor is proposed, with a focus on the validity of the entropy inequality also on the level of the numerical approximation.

In conclusion, we are grateful to all the lecturers, the authors and also the participants for their effort and enthusiasm that led to top quality surveys in the field of mathematical theory for fluid flows on the one hand, and also to the beautiful, friendly and inspiring atmosphere during the school at Kácov. We would also like to use this opportunity to invite all interested researchers to the next, already sixteenth, school that will be held again at Kácov in 2019. Further details will be available from the webpage <http://essam-maff.cuni.cz>, where all links to lecture notes as well as to the history of the School can be found.

Miroslav Bulíček
Eduard Feireisl
Milan Pokorný