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# New Trends and Results in Mathematical Description of Fluid Flows

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*Christian Rohde*

**Fully Resolved Compressible Two-Phase Flow: Modelling, Analytical and Numerical Issues**

New Trends and Results in Mathematical Description of Fluid Flows
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# Preface

This special volume consists of four surveys that are focussed on several aspects in fluid dynamics. The basis for these surveys were series of lectures delivered by Dominic Breit (Heriot-Watt University Edinburgh, United Kingdom), Yann Brenier (Ecole Polytechnique, Palaiseau, France), Pierre-Emmanuel Jabin (University of Maryland, USA) and Christian Rohde (Universität Stuttgart, Germany) at the EMS School in Applied Mathematics *Mathematical Aspects of Fluid Flows* held at Kácov, Czech Republic, May 28–June 2, 2017.

The four surveys cover the following subjects: stochastic partial differential equations in fluid mechanics, different concepts of solutions in ideal fluid mechanics connected with the least action principle, new views on the theory of the transport and continuity equations in connection with the compressible Navier–Stokes equations and related models, and modelling, mathematical and numerical analysis of models as well as the numerical solution of equations describing multi-phase fluid flows.

The main objective of the series of the Kácov Schools is to present new, modern methods, tools and results in the mathematical theory of compressible and incompressible fluids, and more complex systems of partial differential equations. Such a goal must be, however, motivated by studying physically relevant problems. In particular, the rigorous analytical results should lead to good understanding of the behaviour of the model, should predict its limitations and should indicate the kind of numerical method that could be used in order to solve the problem in a stable, accurate and efficient manner. Hand-in-hand with these mathematical properties, the analysis of the models should also lead to the relevant qualitative predictions of the model that are compatible with the physical expectations/experiments. Furthermore the model must obey basic physical principles, such as energy conservation or an entropy inequality. We believe that each of the four surveys fulfills these requirements and presents the most recent results and points of view, and addresses important problems in the mathematical theory of fluids.

One of the popular recent directions is to include suitable small perturbations into the models, which may be of numerical or empirical nature, or may originate in physical uncertainties (such as thermodynamic fluctuations occurring in fluid flows). This issue is discussed in the first part of the volume authored by Dominic Breit. He explains how the classical (deterministic) equations describing fluid flow can be modified to include stochastic effects, leading to stochastic partial differential equations, where all of the quantities involved (such as the density, the velocity and the temperature) are defined on a probability space, and he shows how to include an appropriate Wiener process into the equations. In the introduction the author presents a nice overview of the methods used in the theory of stochastic PDEs. In the rest of the survey, the author discusses two notions of solution: a deterministic equation with random coefficients (semi-deterministic approach) and a fully stochastic problem (finite energy weak martingale solutions), and he highlights the strong and the weak points of both approaches. The available theory for both incompressible and compressible fluids is described in detail. Finally, the main steps of the existence proofs are presented.

Yann Brenier discusses various concepts of generalized and approximate solutions in the mathematical theory of ideal (inviscid) incompressible fluids. The central concept is the relation between solutions of the fluid system in the Eulerian description given by the standard Euler system of partial differential equations and a variational approach based on the least action principle. The least action principle characterizes the admissible trajectories as those minimizing the action functional. It is shown that smooth solutions of the Euler system when restricted to a sufficiently short time interval comply with the least action principle, while they represent critical points of the latter when the time lap is large. Furthermore, various concepts of generalized solutions to the least action principle are introduced and solvability of the latter in this new framework is discussed. Finally, the approximation of the Euler equations, based on a modified least action principle taking into account the energy dissipation, is considered. The intimate relationship of this concept to stochastic and quantum phenomena is discussed in the final part of the chapter.

The next part of the volume is the survey written by Didier Bresch and Pierre-Emmanuel Jabin and is focussed on the regularity and qualitative theory and estimates for the advection equation driven by a nonsmooth velocity field. The authors present a nice overview of the available results, but also show the essential tools and methods used in the most recent results. These are then used to obtain new results for complex systems where the transport equation is coupled to other PDE's, e.g., to the compressible Navier–Stokes system. The aim is also to impose relevant and available estimates for the velocity field or for the unknown, i.e., one cannot impose a bound on the divergence of the velocity or upper or lower bounds on the unknown itself. The whole theory is therefore built for situations when the velocity belongs to a certain Bochner–Sobolev space, which is the natural candidate appearing in the theory of compressible fluids. The survey is split into three main parts: the first one deals with the Lagrangian approach and the logarithmic scale for advective equations; the second part concerns the Eulerian description of the problem and takes into account the power of the technique of renormalized solutions to deduce compactness results for the compressible Navier–Stokes system; the last part provides a beautiful example of how the method introduced by the authors can be applied to the Stokes system coupled with a nonmonotone pressure law.

The last survey, authored by Christian Rohde, ranges from mathematical modelling to sophisticated numerical methods for multi-phase compressible flows. He considers the compressible free flow of homogeneous fluids that occur in liquid and vapour phases and he aims to describe also phase-change phenomena. Two methods in particular are discussed in the survey. First, models which display the phase boundary as a sharp interface are discussed, and second, diffuse interphase models are considered. For both classes of models the associated thermodynamic framework is set up, the applicability of the model is discussed, and finally a suitable numerical scheme is introduced. The most developed among these is the isothermal sharp interphase model, which is understood as a free boundary value problem with appropriate coupling conditions across the interface. These conditions are of the form of an algebraic constitutive relation and the choices of kinetic relations, leading to consistency, well-posedness and a thermodynamic theory, are discussed. Ultimately, these results appear then to be only a weak basis for (multi-dimensional) numerics but it turns out that an appropriate Riemann solver is a key for the construction of moving-mesh finite volume methods for the thermodynamically consistent tracking of interfaces with mass transfer. The sharp interphase model is not able to describe topological changes of the interface. This issue is then solved by using the Navier–Stokes–Korteweg model, where the free energy functional is extended by higher-order terms. The stress tensor is then changed accordingly in order to retain thermodynamical consistency. Finally, a numerical discretization that is able to deal with higher-order and nonlocal terms in the stress tensor is proposed, with a focus on the validity of the entropy inequality also on the level of the numerical approximation.

In conclusion, we are grateful to all the lecturers, the authors and also the participants for their effort and enthusiasm that led to top quality surveys in the field of mathematical theory for fluid flows on the one hand, and also to the beautiful, friendly and inspiring atmosphere during the school at Kácov. We would also like to use this opportunity to invite all interested researchers to the next, already sixteenth, school that will be held again at Kácov in 2019. Further details will be available from the webpage http://essam-maff.cuni.cz, where all links to lecture notes as well as to the history of the School can be found.

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