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Hervé Le Dret

Nonlinear Elliptic Partial Differential Equations

An Introduction

Springer

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In memory of my wife Catherine and of our son Bryan

Preface

This book initially stems from a graduate class I taught at UPMC[1] in Paris between 2004 and 2007. It was first published in French in 2013. I took the opportunity of an English translation to correct the far too many mistakes that had made it through and that I was able to spot this time, to reorganize some parts here and there, to remove several awkward proofs in favor of more streamlined ones, and to add quite a few additional insights and comments. The bulk of the material is, however, essentially the same as that of the 2013 French version, only slightly augmented.

The goal of the book is to present a selection of mathematical techniques that are geared toward solving semilinear and quasilinear partial differential equations and the associated boundary value problems. These techniques are often put to work on examples, and each time, a series of related exercises is proposed. This selection is not meant to be exhaustive by far, nor does it claim to establish a state of the art in the matter. It is conceived more as a basic toolbox for graduate students learning nonlinear elliptic partial differential equations.

The first chapter is a review of results in real and functional analysis, mostly without proofs, although not always, concerning among others integration theory, distribution theory, Sobolev spaces, variational formulations, and weak topologies. It is designed as a sort of vade mecum. The chapter has an appendix that is not required reading for the sequel, but that is meant to satisfy the reader's natural curiosity regarding the somewhat exotic topological vector spaces that tend to crop up in partial differential equation problems.

Chapter 2 is devoted to giving the proofs of the major fixed point theorems: the Brouwer fixed point theorem and the Schauder fixed point theorem. An application of the Schauder fixed point theorem to the resolution of a semilinear partial differential equation is then given.

The focus of Chap. 3 is on superposition operators, which were already introduced in Chap. 2. We study here their continuity, or lack thereof, between various

[1] Now called Sorbonne Université.

function spaces equipped with various topologies. We introduce the concept of Young measures to deal with the case when there is no continuity.

Chapter 4 presents the Galerkin method for solving nonlinear partial differential equations on two examples. The Galerkin method consists in solving finite dimensional approximated problems first and then in passing to the limit when the dimension tends to infinity. The first example is the same semilinear example already solved with fixed point techniques in Chap. 2. The second example is a totally academic example. It is interesting insofar as its nonlinearity shows similarities with that of the stationary Navier-Stokes equations of fluid mechanics.

Chapter 5 is divided into three separate parts. In the first part, we prove several versions of the maximum principle. The second part is a catalogue of elliptic regularity results, mostly without proofs. In the third part, a combination of maximum principle and elliptic regularity is used on an example to prove existence for a semilinear problem via the method of sub- and super-solutions.

We move to an altogether completely different setting in Chap. 6, where we deal with minimization of functionals in the calculus of variations. This is well adapted to solving quasilinear elliptic problems. We consider the scalar case, for which the central idea is convexity. We also treat the vectorial case, which is associated with systems of equations, where subtler convexity variants come into play: quasiconvexity, polyconvexity, and rank-1-convexity.

Chapter 7 offers another take on the calculus of variations, namely the search for critical points of functionals by topological methods. This approach is better suited for semilinear problems, of which we give several examples.

In Chap. 8, we consider quasilinear problems that are not necessarily associated with a functional of the calculus of variations as in Chap. 6. We introduce monotone and pseudo-monotone operators and solve the accompanying variational inequalities. The example of Leray-Lions operators is discussed in detail.

I would like to thank François Murat, whose lecture notes formed the initial basis of the graduate class from which this book evolved in time.

Paris, France Hervé Le Dret
January 2018

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