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Calculus of Variations

Springer

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Preface

The calculus of variations has its roots in the first problems of optimality studied in classical antiquity by Archimedes (ca. 287–212 BC in Syracuse, *Magna Graecia*) and Zenodorus (ca. 200–140 BC). The beginning of the field as a branch of modern mathematics can be traced back to June 1696, when Johann Bernoulli published a description of the *brachistochrone problem* (see Fig. 1.1 on p. 5) and Leonhard Euler's eponymous 1766 treatise *Elementa calculi variationum*.

The field has seen a sweeping revolution since the formulation of David Hilbert's 19th, 20th, and 23rd problems in 1900, which anticipated the modern treatment of minimization problems. This is particularly true for the theory of so-called multiple integrals, that is, integral functionals defined on spaces of vector-valued maps in several variables. Minimization problems for such functionals have been systematically investigated from the 1950s onward, most notably in the works of Charles B. Morrey Jr., Ennio De Giorgi, and John M. Ball. These developments were further fueled by the adaptation of sophisticated mathematical techniques from measure theory, geometric analysis, and the theory of nonlinear PDEs.

On the application side, the discovery of powerful variational principles to investigate questions of material science, in particular in the theories of nonlinear elasticity and microstructure, was (and is) a rich source of challenging problems, which have shaped the field into its modern form. The methods of the modern calculus of variations are now among the most powerful to study highly nonlinear problems in applications from physics, technology, and economics.

The intent of this book is to give an introduction to the classical and modern calculus of variations with a focus on the theory of integral functionals defined on spaces of vector-valued maps in several variables. It leads the reader from the most fundamental results to topics at the forefront of current research. Almost all of the results presented here are not original, but I have reorganized much of the material and also improved some proofs with ideas that were not known when the original arguments were conceived.

This is not an encyclopedic work. While I do aim to show the big picture, many interesting and important results are omitted and often I only present a special case of a more general theorem. Naturally, the choice of topics that I treat in detail is biased by my own personal preferences.

The presentation of the material in this book is based on a few principles:

- Modern techniques are used whenever this leads to a clearer exposition. Most prominently, Young measures are introduced early in the book since they provide a unified and convenient framework to understand a variety of topics.
- I try to use reasonable assumptions, not the most general ones.
- When presented with a choice of how to prove a result, I have usually chosen what is in my opinion the most conceptually clear approach over more elementary ones.
- This book considers minimization problems over vector-valued maps right from the start since this situation has many applications and, in fact, much of the advanced theory was specifically developed for this case.
- Occasionally, I refer to recent theorems without giving a proof. The rationale here is that I want the reader to see the frontier of research without compromising the coherence of the text.
- I include some pointers to the literature and a few (incomplete) historical comments at the end of every chapter.
- The 120 problems are an integral part of the book and I encourage the reader to attempt as many as possible.

This book has two parts: The first seven chapters form the *Basic Course* and are intended to be read in order. They can form the basis of a 30-hour or 40-hour lecture course for an advanced undergraduate or graduate audience (with some selection on the part of the lecturer of what material to cover in detail). In fact, this part is based on lecture notes for the MA4G6 course on the calculus of variations that I lectured at the University of Warwick in 2015 and 2017 (with Richard Gratwick in 2015 and Kamil Kosiba in 2017).

Part II of the book on *Advanced Topics* contains further material that is suitable for a topics course, a reading seminar, or self-study. Here, three themes with only minimal interdependence are covered: rigidity and microstructure in Chapters 8 and 9; linear growth functionals, singularities in measures, and generalized Young measures in Chapters 10–12; and Γ-convergence for sharp-interface limits and homogenization in Chapter 13. Some results presented in these chapters have so far only been accessible in the research literature, and I hope that even seasoned professionals will find something of interest there.

The prerequisites for this book are a good knowledge of functional analysis, measure theory, and some Sobolev space theory. Most of the results that are required throughout the book are recalled in the appendix.

This book is strongly influenced by several previous works. I note in particular the lecture notes on microstructure by Müller [203], Dacorogna's treatise on the calculus of variations [76], Kirchheim's advanced lecture notes on differential inclusions [160], the monograph on Young measures by Pedregal [222], Giusti's introduction to the calculus of variations [137], Dolzmann's book on microstructure in materials [100], as well as lecture notes on several related courses by Jan Kristensen and Alexander Mielke.

I am grateful for any comments, corrections, and suggestions. They can be sent via the book's website, where a list of corrections will also be maintained:

http://www.calculusofvariations.com

comment@calculusofvariations.com

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Coventry, UK — Filip Rindler
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