# An Introduction to the Technique of Formative Processes in Set Theory

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*To Maria Pia and Riccardo,*
*and to the memory of my beloved parents*

*To Melita and Miranda*

# Preface

The decision problem in set theory has been explored intensively in the last decades, mainly with the following goals:

- mechanical formalization of mathematics with a proof verifier based on the set-theoretic formalism [FOS80, OS02, COSU03, OCPS06, SCO11],

- enhancement of the services offered by high-level specification languages including sets and functions among their primitives [SDDS86, KP95],

- handling set constraints [DPPR98, RB15, CRF15] in an advanced declarative programming setting.

This research has generated a large body of results, giving rise to the field of *Computable Set Theory* [CFO89, CF95, COP01, SCO11].

In the design of decision procedures in set theory, one needs inevitably to face the problem of manipulating collections of sets, subject to certain relationships among them that must be maintained invariant. In fact, for a given fragment $\mathfrak{F}$ of set theory of which one is investigating the satisfiability problem, a typical approach consists of finding a *uniform method* that simplifies the satisfying assignments of any given formula in $\mathfrak{F}$, in such a way that some of their representatives—and, possibly, some additional finite structures attached to them—can be confined into a finite collection of candidate assignments that can be effectively generated and tested for satisfaction. As we shall see, this amounts to proving that $\mathfrak{F}$ enjoys a *small model property* or a *small witness-model property.*

This book provides an introduction to the most advanced of these simplification methods developed in computable set theory, namely, the *technique of formative processes.* Each set carries in its extension a complete 'construction plan', which can be conveniently unraveled and manipulated according to one's needs. In the case of a finite number of sets $\mathcal{F}$, which collectively may form a satisfying assignment for a given set-theoretic formula, it proves advantageous to represent $\mathcal{F}$ through its Venn partition, or some of its extensions, rather than by the mere collection of its members. The 'construction plan' $\mathcal{H}_{\mathcal{F}}$ of such an extended Venn partition $\Sigma$ of $\mathcal{F}$ is a *formative process.* By appropriately manipulating $\mathcal{H}_{\mathcal{F}}$, one can produce other formative processes whose induced set collections $\mathcal{F}'$ enjoy the same properties as $\mathcal{F}$, plus possibly additional ones (e.g., rank boundedness or infinite cardinality).

In the mid-1980s, a similar technique began to emerge in the process of solving the satisfiability problem for the fragments of quantifier-free formulae of set theory called MLSP (viz., *Multi-Level Syllogistic with the Powerset operator*; see [CFS85]) and MLSSP (see [Can87] and [Can91]). Specifically, MLSP-formulae are the propositional combinations of atoms of the following types

$$x = y \cup z, \qquad x = y \cap z, \qquad x = y \setminus z, \qquad x \in y, \quad x = \mathcal{P}(y),$$

(where $x, y, z$ stand for set variables, and $\mathcal{P}(\cdot)$ is the powerset operator), whereas MLSSP-formulae are obtained by extending the endowment of MLSP with atoms of the form $x = \{y\}$, involving the singleton operator. This technique was then restated in [CU97, COU02], in much like the current terms, and further refined in [Urs05, CU14], where it was applied to solve the satisfiability problem for the extension MLSSPF of MLSSP with atomic formulae of the form $\mathit{Finite}(x)$, intended to express that $x$ has a finite cardinality. However, unlike MLSSP-formulae, some formulae in MLSSPF are satisfied by infinite assignments only. Hence, the fragment MLSSPF cannot enjoy the *classical* small model property. Instead, the decidability of MLSSPF can be obtained by proving a *small witness-model property*, using the technique of formative processes. Specifically, given an MLSSPF-formula $\Phi$ admitting only infinite satisfying assignments, one can show that the associated formula $\Phi^-$, obtained by dropping from $\Phi$ all literals involving the predicate $\mathit{Finite}(\cdot)$, admits a small satisfying assignment whose formative process can be 'pumped' into a new infinite formative process.

An approach strictly related to the technique of formative processes was also undertaken by Omodeo and Policriti in [OP10, OP12] for the satisfaction problem of the Bernays-Schönfinkel-Ramsey (BSR) class of set-theoretic statements (namely, the class of prenex statements, with a prefix of the form $\exists^*\forall^*$, involving the predicate symbols of equality '$=$' and membership '$\in$' only). Since BSR-statements with a prefix of the form $\exists\exists\forall\forall$ can express infinite sets, as proved in [PP88], the decidability result for the BSR-class has been established by proving for it a small witness-model property. Specifically, it has been shown that any satisfiable BSR-formula $\Phi$ admits a non-necessarily finite model that can be represented by a finite graph of *bounded size* with special nodes, called *rotors*, standing for denumerably infinite families of *well-behaved* sets.

In the form in which they were originally conceived, several decision algorithms in computable set theory are often hard to extend with new set-theoretic constructs that were not built into them from the outset. We expect that a full understanding of the technique of formative processes can enhance its applicability, simplifying combinations of decision procedures, and possibly giving a new research boost to the fascinating area of computable set theory. In fact, the consolidation of the known part of computable set theory is essential not only to promote new discoveries on decidability, but also to convert the theoretical results into technological advances in the field of automated reasoning. For instance, even the most basic layer of automated set reasoning, the so-called *multi-level syllogistic* (MLS), has benefited from being revisited under a tableaux-based approach, which renders its implementation far more efficient (see [CZ00, CZ99] and [COP01, Chapter 14]).

We hope that the present book may contribute to a renewed interest in computable set theory and, specifically, in the related technique of formative processes. We expect that the technique of formative processes might be an essential tool in the solution of some long-standing open problems in the area, such as the satisfiability problem for the extension MLSC of multi-level syllogistic with the Cartesian product. This appears to be particularly relevant insofar as the satisfiability problem for MLSC can be seen, in some sense, as the set-theoretic counterpart of the well-known Hilbert's tenth problem.

**Content:** The book is divided into two parts. The first part is preliminary and consists of three chapters. Chapter 1 introduces the basic set-theoretic terminology and properties used throughout the rest of the book. Chapter 2 is an introduction to the decision problem in set theory. After presenting the syntax and semantics of a comprehensive theory that encompasses all the fragments of set theory considered in the book, the novel notion of *satisfiability by partitions* is introduced. The relations of *simulation* and *imitation* among partitions, which maintain invariant the collection of formulae satisfied by them, are then examined, and a simple case study—the theory BST—is explored in detail. It is proved, in particular, that BST has an NP-complete decision problem. Subsequently, various expressiveness results are obtained and then used to prove two negative results about decidability and *rank dichotomicity.* The latter is a novel concept, strictly related to the small model property. Chapter 3 is devoted to *formative processes.* Preliminarily, formative processes, their supports—*syllogistic boards*—and their applications to the decision problem of MLSP-like fragments are illustrated by means of worked-out examples and pictures. Then all these notions are rigorously presented along with some of their properties. Finally, the so-called *shadowing relationship* among formative processes and its properties are discussed, proving that the final partitions of two shadowing formative processes imitate each other.

The second part of the book consists of two chapters, which are devoted to applications of the technique of formative processes to the decision problems for the theories MLSSP and MLSSPF, respectively. Specifically, given a partition $\Sigma$ and a formative process $\mathcal{H}$ for it, Chapter 4 presents a technique for extracting the subsequence $S_{\mathcal{H}}$ of the *salient steps* of $\mathcal{H}$. The subsequence $S_{\mathcal{H}}$ enjoys the following two properties: (i) its length is bounded by a computable function of the size of the partition $\Sigma$, and (ii) there exists a formative process $\widehat{\mathcal{H}}$ whose sequence of steps is $S_{\mathcal{H}}$ and is such that $\widehat{\mathcal{H}}$ is a shadow process of $\mathcal{H}$. This amounts to showing that MLSSP enjoys the small model property, thereby proving its decidability. In addition, the shadow process $\widehat{\mathcal{H}}$ preserves certain structural properties of $\mathcal{H}$. This fact is exploited in Chapter 5 to prove the existence of suitable infinite shadow processes of $\mathcal{H}$ (*pumping mechanism*). Let $\Phi$ be an MLSSPF-formula and, again, let $\Phi^-$ be the MLSSP-formula obtained by dropping from $\Phi$ all finiteness literals. Then it turns out that one can generate effectively all (descriptions of) infinite processes that are in the shadowing relationship with the formative process of some satisfying assignment for $\Phi^-$ (*small witness-model property*). As proved in Chapter 5, the latter fact yields the decidability of MLSSPF.

All chapters (especially in the first part) contain a number of exercises to help the reader become familiar with the new concepts and techniques. Some exercises simply ask the reader to provide or complete omitted proofs; others are devoted to deepening the comprehension of the topics discussed in the main text. The reader is strongly encouraged to work them out for a full understanding of the concepts and techniques of the book.

Besides some "mathematical maturity," the reader is assumed to have familiarity with the basic set-theoretic apparatus, as well as with some elementary notions from mathematical logic and computability.

Finally, to enhance readability, each proof is terminated by the empty square symbol $\square$, whereas all examples and remarks are closed by the full square symbol $\blacksquare$.

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